S-Noetherian Rings

A Dissertation

Submitted in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

in

Mathematics

by

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May, 2026

Dedication

I dedicate this dissertation to my parents, my supervisor and all my teachers for their support, guidance and encouragement.

UNDERTAKING

I hereby declare that the work presented in this dissertation entitled **"S-Noetherian rings"** submitted to the Department of Mathematics, Chaudhary Mahadeo Prasad Degree College Prayagraj, India. I further declare that this work has not been the basis for the award of any other degree, diploma or any other title elsewhere.

Date:

Place:

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CERTIFICATE

Certified that the work contained in the dissertation entitled **"S-Noetherian rings"** being submitted by **Kuldeep Pathak**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Acknowledgement

I would like to express my sincere gratitude to all those who supported me during the preparation of this dissertation. I express my deep gratitude and sincere thanks to my supervisor, **Dr. Ajim Uddin Ansari**, Assistant Professor, Department of Mathematics, C.M.P. Degree College for excellent supervision, consistent cooperation and continuous encouragement.

I would like to express my deep sense of gratitude to **Prof. Bhupesh Kumar Tripathi**, Head, Department of Mathematics, C.M.P Degree College for their cooperation and providing all necessary facilities during my dissertation work in the department. I extend my sincere thanks to all the faculty members, Department of Mathematics, C.M.P. Degree College (Dr. Anurag Shukla, Dr. Sonika Singh, Dr. Shwetabh Srivastava, Dr. Yadvendra Singh, and Dr. Surya Prakash) for their support and for providing a positive and encouraging academic environment.

I would like to express my sincere thanks to Mr. Aman Pandey, research scholar, Department of Mathematics, C.M.P. Degree College, for his valuable guidance and immense support, and to my friend Arti Yadav for her constant support and helpful discussions, which greatly contributed to the completion of this work.

Finally, I express my deep gratitude to the alimghty GOD and my PARENTS **Sri. Umanath Pathak** (Father) and **Smt. Savita-Pathak** (Mother), for their blessings, constant encouragement, love and care.

Kuldeep Pathak

Abstract

In this dissertation, our aim is to study the concept of S-Noetherian rings, which is a generalization of Noetherian rings. The theory of Notherian rings played an important role in the development of structure theory of commutative rings. Recall that a commutative ring with identity is called Notherian ring if every ideal is finitely genrated. The concept of S-Noetherian rings extends this idea by using a multiplicative subset S of the ring. We begin with some basic preliminaries and recall the definition and properties of Noetherian rings. After that, we study S-Noetherian rings and their basic properties with suitable examples. We also study polynomial rings and formal power series rings over S-Noetherian rings to understand how this property behaves under such extensions. Further, we discuss S-primary decomposition, which is a generalization of the classical primary decomposition.

List of Symbols

R	A commutative ring
I	An ideal of the ring R
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\subseteq$	Subset or equal to
$\subset$	Proper subset
$\in$	Belongs to
$\notin$	Not an element of
$\forall$	For all
$\exists$	There exists
$\Rightarrow$	Implies
$\Leftrightarrow$	If and only if
$R[x]$	Polynomial ring over a field R
$\deg f$	Degree of polynomial f
$\mathrm{rad}(I)$	Radical of I
$\mathrm{Ass}(M)$	Set of all associated primes of M
$\mathrm{Nil}(R)$	Set of all nilpotent elements of R
$\mathrm{Ann}(x)$	Annihilator of x
$Z(R)$	Set of all zero divisors of R
$\cup$	Union of two sets
$\cap$	Intersection of two sets

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Chapter 1

Introduction

The subject of commutative algebra developed mainly during the twentieth century. Many mathematicians contributed to its growth, but one of the most important contributions was given by Emmy Noether. She introduced a new approach to study the rings called Noetherian rings. A ring R is called Noetherian if each ideal of R is finitely generated. The Noetherian property is also closely related to the ascending chain condition on ideals. According to this condition, if we take a sequence of ideals such that$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$, then there exists $k \in \mathbb{N}$ such that $I_k = I_n$ for all $n \geq k$. This property is very useful because it prevents infinite growth of ideals and gives better control over the structure of the ring. Noetherian rings are very important in algebra because many strong and useful results hold in this setting. One of the most famous results is the Hilbert Basis Theorem which states that if a ring is Noetherian, then its polynomial ring and formal Power series ring are also Noetherian. This result is very important because polynomial rings and formal power series rings are used in many areas of mathematics, especially in commutative algebra and algebraic geometry. These rings are useful in local algebra, where we study the behavior of algebraic objects near a particular point.

Another important concept in commutative algebra is primary decomposition. It allows us to express an ideal as an intersection of finitely many primary ideals. This helps in understanding the structure of ideals in a

clearer and more detailed way.

It has been observed that many rings do not satisfy the Noetherian condition, yet they still possess several useful properties similar to those of Noetherian rings. Because of this reason, mathematicians began to look whose class is broder than that of Noetherian ring for more general types of rings. As one of its crucial generalization, Anderson and Dumitrescu [1] introduced the concept of S-Notherian rings. Let R be a commutative ring with identity and S be a multiplicatively closed subset of R. An ideal I of R is called S-finite if there exist $s \in S$ and a finitely generated ideal F of R such that $sI \subseteq F \subseteq I$. Also R is called S-Notherian ring if each ideal of R is S-finite.

The motivation behind studying S-Noetherian rings is to understand rings which are not Noetherian globally but show Noetherian-like behavior locally. They have transferred several results on Noetherian rings to S-Noetherian rings. Thereafter S-Noetherian rings are continuously studied by many authors ([4], [5], [6], [12], [13], [22]).

In this dissertation, we study Noetherian rings and S-Noetherian rings in a simple and systematic way. The main aim of this work is to understand the structure of these rings and to see how the idea of finiteness affects their properties. This dissertation has been divided into four chapters. Each chapter is divided into a number of sections. The **present** chapter is the introduction. The **second** chapter is preliminaries, in which we recall some basic definitions and results to be used in the remaining chapters. The **third** chapter deals with the study of Noetherian rings. In this chapter, we study several properties and characterizations of Noetherian rings. For instance, Cohen's theorem, Hilbert Basis Theorem for R[x] and R[[x]], and Lasker–Noether theorem.

The **fourth** chapter deals with the study of S-Noetherian rings. First, we provide the definition of S-Noetherian rings with several interesting examples. After that, we provide several properties and characterizations of this class of rings. Among other results, we provide S-versions of Cohen's theorem and Hilbert Basis Theorem for R[x] and R[[x]]. We end the

chapter by studying the S-version of the classical Lasker–Noether theorem.

Chapter 2

Preliminaries

In this chapter, we state some known results and give definitions with examples which will be need in subsequent chapters. Most of the results given here can be found in [2], [10], [19], [22]. Throughout the dissertation all rings are assumed to be commutative rings with identity unless otherwise stated.

Definition 2.1. *A ring $(R, +, \cdot)$ is a non-empty set R equipped with two binary operations $+$ and $\cdot$ such that:*

1. *$(R, +)$ is an abelian group*
2. *$(R, \cdot)$ is a semigroup*
3. *Multiplication distributes over addition, i.e., $x(y+z) = xy+xz$ and $(x+y)z = xz + yz$ for all $x, y, z \in R$.*

Further, if $xy = yx$ for all $x, y \in R$, then R is called a ***commutative ring****.*

Definition 2.2. *A ring R is said to be a* ***ring with identity*** *if there exists an element $1 \in R$, $1 \neq 0$, such that $1a = a1 = a \quad$ for all $a \in R$.*

Example 2.1. *$(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are commutative rings with identity.*

Definition 2.3. *Let R be a ring with identity. Then an element $a \in R$, $a \neq 0$, is called* ***unit*** *if there exists $b \in R$ such that $ab = ba = 1$.*

Definition 2.4. *Let R be a ring. Then a non-zero element $a \in R$ is called a* ***zero divisor*** *if there exists non-zero $b \in R$ such that $ab = 0$ or $ba = 0$.*

Example 2.2. *In $\mathbb{Z}_6$, $\bar{2} \cdot \bar{3} \equiv 0$, $\bar{3} \cdot \bar{4} \equiv 0$ Hence $\bar{2}, \bar{3}, \bar{4}$ are zero divisors in $\mathbb{Z}_6$*

Definition 2.5. *A commutative ring R with identity is called an* ***integral domain*** *if for $a, b \in \mathbb{R}$, whenever $ab = 0$, then either $a = 0$ or $b = 0$.*

Definition 2.6. *Let R be a ring then an element $a \in R$ is called* ***idempotent*** *if $a^2 = a$.*

Definition 2.7. *A ring R is called a* ***Boolean ring*** *if every element of R is idempotent.*

Definition 2.8. *Let R be a ring. Then an element $a \in R$ is called* ***nilpotent*** *if $a^n = 0$ for some $n \in \mathbb{N}$.*

Definition 2.9. *A commutative ring R with identity is called a* ***field*** *if every non-zero element of R is a unit.*

Example 2.3. *$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.*

Definition 2.10. *Let R be a ring. Then a non-empty subset $S \subseteq R$ is called a* ***subring*** *of R if:*

1. *S is closed under subtraction and multiplication*
2. *S forms a ring under the operations of R.*

Example 2.4. *$2\mathbb{Z}$ is a subring of $\mathbb{Z}$.*

Definition 2.11. *Let R be ring then a subset $I \subseteq R$ is called a* ***ideal*** *if:*

1. *$(I, +)$ is a subgroup of $(R, +)$*
2. *$rx \in I$ for all $r \in R$ and $x \in I$.*

Definition 2.12. *Let R be a ring and $S \subset R$. Then the ideal* ***generated by*** *S is denoted by $\langle S \rangle$ which is defined as the smallest ideal of R containing S. Further, if $S = \{a\}$ for some $a \in R$, then $\langle a \rangle$ is called a* ***principal Ideal.***

Definition 2.13. *(**Maximal Ideal**) Let R be a ring and I be an ideal of R and $I \neq R$. We say that I is a maximal ideal if whenever $I \subset J \subset R$, Then either $I = J$ or $J = R$. In other words, a maximal ideal is not properly contained in any other proper ideal of R.*

Definition 2.14. *(**Prime Ideal**) Let R be a ring and P be an ideal of R and $P \neq R$. We say that P is a prime ideal if $ab \in P$ then either $a \in P$ or $b \in P$.*

Example 2.5. *In $\mathbb{Z}$, the ideal $p\mathbb{Z}$ is prime if and only if p is a prime integer. In $\mathbb{Z}$, $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$.*

Definition 2.15. *Let R be a ring. Then an Ideal $I \subset R$ is* ***finitely generated*** *if there exist $a_1, a_2, \ldots, a_n \in R$ such that $I = (a_1, a_2, \ldots, a_n)$, i.e., $I = \{r_1a_1 + r_2a_2 + \cdots + r_na_n \ : \ r_i \in R\}$ and, an ideal $I \subset R$ is* ***infinitely generated*** *if it is not finitely generated, i.e., there is no finite set $a_1, a_2, \ldots, a_n \in R$ such that $I = (a_1, \ldots, a_n)$*

Example 2.6. *In $\mathbb{Z}$, every ideal is finitely generated.*

Example 2.7. *Let the Polynomial ring $K = [x_1, x_2, \ldots]$ and $I = (x_1, x_2, \ldots)$ be an ideal of k. Then I is infinitely generated.*

Definition 2.16. *(**Ring Homomorphism**) A ring homomorphism is a map $f : R \to T$ such that*

1. *$f(x + y) = f(x) + f(y)$*
2. *$f(xy) = f(x)f(y)$, for all $x, y \in R$.*

Further, if f is injective (resp., surjective), then f is called a ***monomorphism****(resp,* ***epimorphism****). Also, if f is bijective, then it is called an* ***isomorphism****.*

Definition 2.17. *Let $f : R \to T$ be a ring homomorphism. Then* ***kernel of*** *f is denoted by $ker f$ and defined by* $\ker f = \{x \in R \mid f(x) = 0\}$.

Proposition 2.1. $\ker f$ *is an ideal of R. Also f is injective if and only if* $\ker f = \{0\}$.

Definition 2.18. *(**Image of a Homomorphism**) Let $f : R \to T$ be a ring homomorphism then the image of f is defined as $Im(f) = \{f(x) \mid x \in R\}$. The image of a ring homomorphism is a subring of T.*

Theorem 2.1. *(**Characterization of Prime Ideal**) Let R be a commutative ring with identity and let P be an ideal of R such that $P \neq R$. Then P is a prime ideal if and only if the quotient ring R/P is an integral domain.*

Theorem 2.2. *(**Characterization of Maximal Ideal**) Let R be a commutative ring with identity and let M be an ideal of R. Then M is a maximal ideal if and only if the quotient ring R/M is a field.*

Theorem 2.3. *(**First Isomorphism Theorem**) Let $f : R \to T$ be a ring homomorphism. Then* $R/\ker f \cong \mathrm{Im}(f)$.

Theorem 2.4. *(**Correspondence Theorem**) Let $f : R \to T$ be a surjective ring homomorphism. Then there exists a one-to-one correspondence between the ideals of R containing* $\ker f$ *and the ideals of T. Moreover, this correspondence preserves inclusion.*

Chapter 3

Noetherian Rings

In this chapter, we study basic properties and characterization of Noetherian rings. Most of the results can be found in ([2], [10], [19], [22]).

3.1 Properties and characterizations

We begin this section by providing the definition of Noetherian ring with a number of interesting examples.

Definition 3.1. *A ring R is said to be Noetherian ring if every ideal of R is finitely generated.*

Example 3.1. *Every field is Noetherian. because in a field F, the only ideals are (0) and F. In particular, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are Noetherian rings.*

Example 3.2. *Every PID (Principal Ideal Domain) is Noetherian, because in a PID, every ideal is generated by single element. In particular, ring of integers $\mathbb{Z}$ is Noetherian.*

Example 3.3. *Let $R = \mathbb{Q}[x_1, x_2, \ldots, x_n, \ldots]$ be the polynomial ring in infinitely many variables. Take $I = \langle x_1, x_2, \ldots, x_n, \ldots \rangle$ then I is not finitely generated, so R is not Noetherian.*

Example 3.4. *Every finite commutative ring is Noetherian since all ideals are finite and hence finitely generated. In particular, $\mathbb{Z}_{\mathbb{M}}$ is a Noetherian ring.*

Example 3.5. *Let p be a prime number and consider the ring*

$$R = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}.$$

Then this R is not Noetherian. Indeed, consider the ideal

$$I = \{(a_1, a_2, a_3, \dots) \in R \mid a_n = 0 \text{ for all but finitely many } n\}.$$

This ideal is not finitely generated; hence R is not Noetherian.

Example 3.6. *Let (p_n) be the sequence of prime numbers and define*

$$R = \prod_{n=1}^{\infty} \mathbb{F}_{p_n},$$

where $\mathbb{F}_{p_n}$ denotes the finite field with p_n elements. Then this ring R is not Noetherian, because there exists an infinite strictly increasing chain of ideals

$$\mathbb{F}_{p_1} \times 0 \times 0 \times \cdots \subseteq \mathbb{F}_{p_1} \times \mathbb{F}_{p_2} \times 0 \times \cdots \subseteq \cdots$$

which does not stabilize. Hence R does not satisfy the ascending chain condition on ideals.

Theorem 3.1. *Let R be a ring. The following statements are equivalent:*

1. *Every ideal of R is finitely generated.*
2. *Every ascending chain of ideals of R is stationary.*
3. *Every non-empty set of ideals of R has a maximal element.*

Proof. **(1) implies (2)** Assume every ideal of R is finitely generated. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a chain of ideals of R. Define $I = \bigcup_{n=1}^{\infty} I_n$. Then I is an ideal of R. Since R is Noetherian, I is finitely generated. So suppose $I = (a_1, a_2, \dots, a_k)$. Suppose each a_i belongs to some I_{n_i}. Let $N = \max\{n_1, n_2, \dots, n_k\}$. Then all generators belong to I_N, so $I \subseteq I_N$.

And we know $I_N \subseteq I$ always holds. Therefore $I_N = I$. Hence the ascending chain of ideals in R is stationary.

(2) implies (3) Assume that the ascending chain of ideals in R is stationary. Suppose S be a non-empty set of ideals. Choose any ideal $I_1 \in S$. If I_1 is not maximal, then there exists $I_2 \in S$ such that $I_1 \subset I_2$. If I_2 is not maximal, then there exists $I_3 \in S$ such that $I_2 \subset I_3$.Continuing this process, we get an ascending chain $I_1 \subset I_2 \subset I_3 \subset \cdots$ Since ascending chain is stationary, this process must stop. Hence some I_n is maximal in S.

(3) implies (1) Assume every non-empty set of ideals has a maximal element. Let I be an ideal of A. Consider the set

$$F = \{J \subseteq I \mid J \text{ is finitely generated}\}$$

. This set is non-empty because $\{0\} \in F$. By assumption, F has a maximal element, say J.We claim that $J = I$. Suppose $J \neq I$, then there exists $a \in I \setminus J$. Consider the ideal $J' = J + (a)$.Then $J' \subseteq I$ and is finitely generated. But $J \subset J'$, contradicting maximality of J. Hence $J = I$. Therefore I is finitely generated. □

Proposition 3.1. *If R is Noetherian and $\phi : R \to B$ is a surjective ring homomorphism, then B is Noetherian.*

Proof. Since ϕ is a surjective homomorphism, by the fundamental theorem of homomorphism, $R/\ker(\phi) \cong B$. Let I be an ideal of B. Then its preimage in R is an ideal containing $\ker(\phi)$. Because R is Noetherian, every ideal of R is finitely generated. Hence every ideal of $R/\ker(\phi) \cong B$ is finitely generated. Thus B is Noetherian.

□

Proposition 3.2. *Let R be a subring of B. Suppose R is Noetherian and B is finitely generated as an R-module. Then B is a Noetherian ring.*

Proof. Since B is a finitely generated R-module and R is Noetherian, B is Noetherian as an R-module. Hence every submodule of B is finitely

generated. But ideals of B are also R-submodules of B. Hence every ideal of B is finitely generated. Therefore, B is Noetherian. □

Proposition 3.3. *If R is Noetherian and S is any multiplicatively closed subset of R, then $S^{-1}R$ is Noetherian.*

Proof. Suppose J is an ideal of $S^{-1}R$. We show J is finetly generated. Define $I = \{\, a \in R \mid a/1 \in J \,\}$. Then I is an ideal of R. Since R is Noetherian, so every ideal of A is finetly genrated. So $I = (a_1, \ldots, a_n)$ is finetly generated for some $a_1, \ldots, a_n \in R$. Now we claim that $J = (a_1/1, \ldots, a_n/1)$. Take any element $\frac{a}{s} \in J$. Because $a/1 \in J$, by definition of I, $a \in I$ Since $I = (a_1, \ldots, a_n)$, we have $a = r_1a_1 + \cdots + r_na_n$ for some $r_i \in R$. Now divide by s,

$$\frac{a}{s} = \frac{r_1}{s}\frac{a_1}{1} + \cdots + \frac{r_n}{s}\frac{a_n}{1}.$$

This shows $\frac{a}{s}$ is in the ideal generated by $(a_1/1, \ldots, a_n/1)$. Hence Every element of J lies in that ideal so $J = (a_1/1, \ldots, a_n/1)$. Hence J is finitely generated and J is arbitrary. Hence every ideal of $S^{-1}R$ is finitely generated. Hence $S^{-1}R$ is Noetherian.

□

Corollary 3.1. *If R is Noetherian and P is a prime ideal of R, then R_P is Noetherian.*

Proof. Since R_P is the localization of R at the multiplicatively closed set $S = R \setminus P$, the result follows from Proposition 3.3 .

□

Example 3.7. *Let $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : p \nmid b,\ a, b \in \mathbb{Z} \right\}$ is a localization of $\mathbb{Z}$ as $S = \mathbb{Z} - p\mathbb{Z}$. Then $\mathbb{Z}_{(p)}$ is the Noetherian ring as $\mathbb{Z}$ is Noetherian, and (by Prepositions 3.3) we know that the localization of Noetherian ring is Noetherian.*

Theorem 3.2. ***(Cohen's Theorem)*** *Let R be a ring. Then R is Noetherian if and only if every prime ideal of R is finitely generated.*

Proof. If R is Noetherian, then every ideal of R is finitely generated. In particular, every prime ideal of R is finitely generated.

Conversely, suppose that every prime ideal of R is finitely generated. We prove that R is Noetherian. Assume, to the contrary, that R is not Noetherian. Then there exists an ideal of R which is not finitely generated. Let $\mathcal{S} = \{I \subseteq R \mid I \text{ is not finitely generated.}\}$ Then $\mathcal{S}$ is nonempty. Partially order $\mathcal{S}$ by inclusion. Let $\{I_\lambda\}$ be a chain in $\mathcal{S}$. Then $I = \bigcup I_\lambda$ is an ideal of R. If I were finitely generated, say $I = (a_1, \ldots, a_n)$, then each a_i would belong to some I_{λ_i}. Since the set $\{I_\lambda\}$ is a chain, there exists I_{λ_0} containing all a_i, so $I \subseteq I_{\lambda_0}$, a contradiction. Hence I is not finitely generated. Thus every chain in $\mathcal{S}$ has an upper bound. By Zorn's Lemma, there exists a maximal element M in $\mathcal{S}$. Then M is not finitely generated, but every ideal properly containing M is finitely generated. We claim that M is prime. Suppose $ab \in M$ and $a \notin M$. Then the ideal $M + (a)$ properly contains M, hence it is finitely generated. Also, the ideal$(M : a) = \{r \in R \mid ra \in M\}$ properly contains M, and hence is finitely generated. From this it follows that M is finitely generated, a contradiction. Therefore M is prime. But M is a prime ideal which is not finitely generated, contradicting our assumption. Hence R must be Noetherian. □

3.2 Polynomial ring over Noetherian ring

A polynomial ring, denoted by $R[x]$, is the set of all polynomials

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

with coefficients a_i from a ring R and indeterminate x. It forms a ring under the usual operations of polynomial addition and multiplication.

If R is a commutative ring with unity, then $R[x]$ is also a commutative ring with unity.

A polynomial ring can be extended to multiple variables, denoted

$$R[x_1, x_2, \ldots, x_n].$$

Example 3.8. *Let $R = \mathbb{Z}$, the ring of integers. Then $\mathbb{Z}[x]$ is the set of all polynomials with integer coefficients.*

For instance, consider the polynomials

$$f(x) = 2 + 3x + x^2 \quad \textit{and} \quad g(x) = 1 + 4x.$$

Addition:

$$f(x) + g(x) = (2 + 1) + (3x + 4x) + x^2 = 3 + 7x + x^2.$$

Multiplication:

$$f(x) \cdot g(x) = (2 + 3x + x^2)(1 + 4x).$$

Expanding,

$$= 2(1 + 4x) + 3x(1 + 4x) + x^2(1 + 4x)$$

$$= 2 + 8x + 3x + 12x^2 + x^2 + 4x^3$$

$$= 2 + 11x + 13x^2 + 4x^3.$$

Thus, $\mathbb{Z}[x]$ forms a polynomial ring.

Theorem 3.3. *(**Hilbert Basis Theorem**) If R is a Noetherian ring , then the polynomial ring $R[X]$ is also Noetherian.*

Proof. If $R = 0$, then result is true, so we assume $R \neq \{0\}$. To prove each ideal I in $R[X]$ is finitely generated, we assume I is not finitely generated and will get a contradiction. We have $I \neq (0)$. Define a sequence of polynomials $f_1, f_2, \ldots$ in I as follows.

(1) Pick f_1 to be an element of $I - (0)$ with minimal degree. (It is not

unique.)

(2) Since $I \neq (f_1)$, as I is not finitely generated, pick f_2 in $I - (f_1)$ with minimal degree. Note $\deg f_1 \leq \deg f_2$ by the minimality condition on $\deg f_1$.

(3) For $k \geq 2$, if we have defined $f_1, \ldots, f_k$ in I then $I \neq (f_1, \ldots, f_k)$ since I is not finitely generated, so we may pick f_{k+1} in $I - (f_1, \ldots, f_k)$ with minimal degree.

We have $\deg f_k \leq \deg f_{k+1}$ for all k: the case $k = 1$ was checked before, and for $k \geq 2$, f_k and f_{k+1} are in $I - (f_1, \ldots, f_{k-1})$ so $\deg f_k \leq \deg f_{k+1}$ by the minimality condition on $\deg f_k$.

For $k \geq 1$, let $d_k = \deg f_k$ and c_k be the leading coefficient of f_k, so $d_k \leq d_{k+1}$ and $f_k(X) = c_k X^{d_k} +$ lower-degree terms. The ideal $(c_1, c_2, \ldots)$ in R (an ideal of leading coefficients) is finitely generated since R is Noetherian. Each element in this ideal is an R-linear combination of finitely many c_k, so $(c_1, c_2, \ldots) = (c_1, \ldots, c_m)$ for some m. Since $c_{m+1} \in (c_1, c_2, \ldots, c_m)$, we have

$$c_{m+1} = \sum_{k=1}^{m} r_k c_k \tag{3.1}$$

for some $r_k \in R$. From the inequalities $d_k \leq d_{m+1}$ for $k \leq m$, the leading term in $f_k(X) = c_k X^{d_k} + \cdots$ can be moved into degree d_{m+1} by using $f_k(X) X^{d_{m+1} - d_k} = c_k X^{d_{m+1}} + \cdots$, and this is in I since $f_k(X) \in I$ and I is an ideal in $R[X]$. By (2.1), the R-linear combination $\sum_{k=1}^{m} r_k f_k(X) X^{d_{m+1} - d_k}$ is in the ideal $(f_1, \ldots, f_m)$ and its coefficient of $X^{d_{m+1}}$ is $\sum_{k=1}^{m} r_k c_k$, which equals the leading coefficient c_{m+1} of $f_{m+1}(X)$ in degree d_{m+1}. The difference

$$f_{m+1}(X) - \sum_{k=1}^{m} r_k f_k(X) X^{\deg f_{m+1} - \deg f_k} \tag{3.2}$$

is in I, it is not 0 since $f_{m+1} \in I - (f_1, \ldots, f_m)$, and it has degree less than d_{m+1} since the terms $c_{m+1} X^{d_{m+1}}$ cancel out. But $f_{m+1}(X)$ has minimal degree among polynomials in $I - (f_1, \ldots, f_m)$, and (3.2) is in $I - (f_1, \ldots, f_m)$ with lower degree than d_{m+1}. That's a contradiction. Thus I is finitely generated. $\square$

Corollary 3.2. *If R is Noetherian, then $R[x_1, \ldots, x_n]$ is Noetherian.*

Proof. For $n = 1$, by Hilbert's basis theorem, since R is Noetherian, the polynomial ring $R[x_1]$ is Noetherian. Assume that $R[x_1, \ldots, x_k]$ is Noetherian for some $k \geq 1$. Now$R[x_1, \ldots, x_k, x_{k+1}] = (R[x_1, \ldots, x_k])[x_{k+1}]$. Since $R[x_1, \ldots, x_k]$ is Noetherian by the induction hypothesis, applying Hilbert's basis theorem gives that$(R[x_1, \ldots, x_k])[x_{k+1}]$ is Noetherian. Hence

$$R[x_1, \ldots, x_k, x_{k+1}]$$

is Noetherian. Hence by induction, if R is Noetherian then $R[x_1, x_2, \ldots, x_n]$ is Noetherian.

□

Example 3.9. *Since $\mathbb{Z}$ is a PID and hence Noetherian, $\mathbb{Z}[x]$ is Noetherian by Hilbert basis theorem, but $\mathbb{Z}[x]$ is not a principal ideal domain, e.g. $(2, x)$ is not a principal ideal).*

Example 3.10. *If F is a field, then F is Noetherian and so by using Hilbert basis theorem $F[x]$ is also Noetherian.*

Example 3.11. ***(Affine algebra)***$R = K[x_1, x_2, \ldots, x_n]/I$ *is always a Noetherian ring, where I is an ideal of $K[x_1, x_2, \ldots, x_n]$. In particular, $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is a Noetherian ring.*

3.3 Formal power series over Noetherian ring

A **formal power series** is a special kind of formal series, of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, where the a_n, called coefficients, are numbers or, more generally, elements of some ring, and the x^n are formal powers of the symbol x that is called an indeterminate or, commonly, a variable. A formal power series with coefficients in a ring R is called a

formal power series over R. The formal power series over a ring R form a ring, commonly denoted by $R[[x]]$.

Theorem 3.4. *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$, it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be an ideal. We have to show that I is a finitely generated ideal.For each integer d denote$I_d = \{\, a \in R \mid ax^d + \text{higher order term } \in I\}$. Then we see that $I_0 \subset I_1 \subset \cdots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \cdots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d), \qquad j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{higher order term}$ then $I_d = (a_{d,j})$. Denote $I' = \big(\{f_{d,j}\}_{d=0,\ldots,d_0;\, j=1,\ldots,n_d}\big)$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I$. Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I$. Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that $f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I$, and so on. In the end we see that $f = \sum c_{d,i} f_{d,i} + \sum_i \left(\sum_e c_{i,e} x^e\right) f_{d_0,i}$ is contained in I' as desired. □

3.4 Existence of primary decomposition

The decomposition of an ideal is a traditional pillar of ideal theory. In this section we study primary ideals and Lasker Noether theorem for Noetherian ring.

Definition 3.2. ***(Primary ideal)*** *A proper ideal Q in ring R is called primary ideal if $ab \in Q$ and $a \notin Q \;\Rightarrow\; b^n \in Q$ for some $n \in \mathbb{N}$. Equivalently, $ab \in Q$ and $a \notin Q \;\Rightarrow\; b \in rad(Q)$*

Properties 3.1. *Every prime ideal is primary.*

Proof. If $ab \in P$ and $a \notin P$ then $b \in P$. So P is primary. □

Properties 3.2. *The ideal Q is primary iff every zero divisor in R/Q is nilpotent.*

Proof. Let Q is primary. Let $a+Q \in R/Q$ is a zero divisor. Hence $\exists\, b+Q \in R/Q, b \notin Q$ such that (a + Q)(b + Q) = Q This implies that $a \cdot b + Q = Q$ hence $ab \in Q$ and $b \notin Q$ This implies that $a^n \in Q \quad$ for some $n \in \mathbb{N}$ (as Q is primary) Thus $a^n + Q = Q$ This implies that$(a + Q)^n = Q$ hence $a + Q$ is nilpotent in R/Q.

Conversely, assume that every zero divisor of R/Q is nilpotent. Let $a, b \in R$ such that $ab \in Q$ and $a \notin Q$. Then $a + Q \neq Q$ and

$$(a + Q)(b + Q) = ab + Q = Q.$$

. Hence $b + Q$ is a zero divisor in R/Q. By assumption, every zero divisor in R/Q is nilpotent. Therefore, $b + Q$ is nilpotent. So there exists some $n \in \mathbb{N}$ such that $(b + Q)^n = Q$. This implies $b^n + Q = Q$, and hence $b^n \in Q$. Thus, whenever $ab \in Q$ and $a \notin Q$, we get $b^n \in Q$ for some $n \in \mathbb{N}$. Therefore, Q is a primary ideal. □

Properties 3.3. *If Q is primary ideal then* rad(Q) *is prime ideal and is the unique smallest prime ideal containing Q. But converse need not be true.*

Proof. Let $ab \in \text{rad}(Q)$ this implies that $(ab)^n \in Q \quad$ for some $n \in \mathbb{N}$ i.e. $a^n b^n \in Q$ this implies that either $a^n \in Q$ or $(b^n)^m \in Q \quad$ for some $m \in \mathbb{N}$ this implies that either $a \in \text{rad}(Q)$ or $b^k \in Q,\ k = m \cdot n$, thus either $a \in \text{rad}(Q)$ or $b \in \text{rad}(Q)$ So radical of primary ideal is prime.

□

Properties 3.4. *If Q is an ideal whose radical is a maximal ideal then Q is a primary ideal.*

Proof. We show that every zero divisor in R/Q is nilpotent to prove Q is primary. Since we know that $\text{nil}(R/I) = \sqrt{I}/I$ also given that $\sqrt{Q} = M,$ a maximal ideal So $\text{nil}(R/Q) = \sqrt{Q}/Q = M/Q$ Since we know that nil(A) = intersection of all prime ideals So R/Q has unique prime ideal M/Q (because M is maximal ideal of R). This implies that M/Q is unique maximal ideal. This implies that $R/Q - M/Q$ is unit Also since

element of M/Q is nilpotent as $M/Q = \text{nil}(R/Q)$ So every element in R/Q is either unit or nilpotent. This implies that every zero divisor of R/Q is nilpotent. Hence Q is primary ideal.

□

Properties 3.5. *Suppose M is a maximal ideal and Q is an ideal with $M^n \subseteq Q \subseteq M$ for some $n \geq 1$ Then Q is primary with $\sqrt{Q} = M$.*

Proof. Suppose $M^n \subseteq Q \subseteq M$ for some $n \geq 1$, where M is maximal ideal (so prime). Since $Q \subseteq M$ this implies that $\text{rad}(Q) \subseteq \text{rad}(M)$ because $\text{rad}(P) = P$ for prime i.e. $\text{rad}(Q) \subseteq M$ Also since $M^n \subseteq Q$ for some $n \geq 1$ this implies that $\text{rad}(M^n) \subseteq \text{rad}(Q)$ this implies that $\text{rad}(M) \subseteq \text{rad}(Q)$ (because $\text{rad}(I^n) = \text{rad}(I)$). This implies that $M \subseteq \text{rad}(Q)$. Hence, $\sqrt{Q} = M$ So by (Property 3.4), Q is primary ideal. □

Note: It is not true for prime ideal.

Example 3.12. *Consider the ring of integers $\mathbb{Z}$ and the ideal $Q = (4)$. Then the ideal $Q = (4)$ is a primary ideal. As suppose $ab \in (4)$. Then by definition of primary ideal either $a \in (4)$, or $b^n \in (4)$ for some positive integer n. If we take $b = 2$ and $n = 2$, then $b^n \in (4)$. Thus, (4) is a primary ideal in $\mathbb{Z}$.*

Definition 3.3. ***(Irreducible ideal)*** *A proper ideal I in the ring R is said to be irreducible, if I cannot be written non-trivially as the intersection of two other ideals, i.e., if $I = J \cap K$ with ideals $J \quad K$, then either $I = J$ or $I = K$.*

Example 3.13. *All prime ideal of ring R is irreducible.*

Example 3.14. *In $\mathbb{Z}$, $p\mathbb{Z}$ are irreducible.*

Example 3.15. *$I = (x^2, xy)$ in $k[x, y]$, then $I = (x) \cap (x^2, y)$. So I is not irreducible.*

Example 3.16. ***(Primary ideal need not be irreducible)*** *for example* $(x, y)^2$ *is Primary ideal which is not irreducible.* $(x, y)^2 = (x, y^2) \cap (y, x^2)$ *as* $(x) + (x, y)^2 = (x, y^2)$ *&* $(y) + (x, y)^2 = (y, x^2)$.

In a Noetherian ring, however irreducible ideals are necessarily Primary.

Lemma 3.1. *In a Noetherian ring* R*, every ideal is a finite intersection of irreducible ideals.*

Proof. Suppose the statement is false. Consider the set

$$\Sigma = \{I \subset R \mid I \text{ is not a finite intersection of irreducible ideals}\}.$$

If $\Sigma \neq \emptyset$, since R is Noetherian, the set of ideals satisfies ACC. Hence Σ contains a maximal element, say a. Since $a \in \Sigma$, it is not irreducible. Therefore, $a = b \cap c$ with $b \supset a$, $c \supset a$. By maximality of a, both b and c are finite intersections of irreducible ideals. Hence a is also a finite intersection of irreducible ideals, a contradiction. □

Now, we prove existence of primary decomposition in Noetherian ring.

Theorem 3.5. ***(Laskar-Noether theorem)*** *Let* R *be a Noetherian ring. Then*

(1) Every irreducible ideal is primary.

(2) Every proper ideal in R *is a finite intersection of irreducible ideals.*

Proof. (1) Let Q be an irreducible ideal and suppose $ab \in Q$ and $b \notin Q$. For each fixed $n > 0$, consider $A_n = \{x \in R \mid a^n x \in Q\}$. Then A_n is an ideal of R. Clearly, $A_1 \subseteq A_2 \subseteq \cdots$. Since R is Noetherian, this ascending chain must stabilize, i.e., there exists $k \in \mathbb{N}$ such that $A_k = A_{k+1} = \cdots$ Consider the two ideals

$$I = (a^k) + Q \quad \text{and} \quad J = (b) + Q.$$

We show that $Q = I \cap J$. Clearly $Q \subseteq I$ and $Q \subseteq J$, hence $Q \subseteq I \cap J$. If $y \in I \cap J$, then $y \in I$ and $y \in J$. So $y = a^k z + q$ for some $z \in R$, $q \in Q$.

Since $ab \in Q$, we have $aJ \subseteq Q$. In particular, as $y \in J$, we get $ay \in Q$. Then

$$a^{k+1}z = a(a^k z) = a(y - q) = ay - aq \in Q.$$

Because $ay \in Q$ and $aq \in Q$. Hence $z \in A_{k+1} = A_k$, so $a^k z \in Q$. Therefore $y = a^k z + q \in Q$. Thus $I \cap J \subseteq Q$ and hence $Q = I \cap J$. Now since Q is irreducible and $J = (b) + Q \neq Q$ (as $b \notin Q$), we must have $I = (a^k) + Q = Q$. Hence $a^k \in Q$. Thus Q is primary.

(2) Let S be the collection of ideals of R that cannot be written as a finite intersection of irreducible ideals. We wish to show that $S = \varnothing$. If$S \neq \varnothing$, then there is a maximal element I in S, since R is Noetherian. Then I is not irreducible. Hence there exist ideals J and K such that

$$I = J \cap K, \quad I \neq J, \quad I \neq K$$

Thus $I \subset J$ and $I \subset K$. Since I is maximal in S, neither J nor K belongs to S. Hence J and K can be written as finite intersections of irreducible ideals. Therefore $I = J \cap K$ can also be written as a finite intersection of irreducible ideals, a contradiction. Hence $S = \varnothing$. Thus every proper ideal in R is a finite intersection of irreducible ideals.

□

Chapter 4

S-Noetherian Rings

In 2002, Anderson and Dumitrescu [1] introduced the notions of S-Noetherian rings as a generalization of Noetherian rings. In this chapter, we study S-Noetherian rings , their properties and characterizations.

4.1 Definitions and examples

Definition 4.1. *(S-**finite Ideal**) Let R be a ring and $S \subseteq R$ a multiplicative set. An Ideal I of R is said to be S-finite if $\exists s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$.*

Definition 4.2. *(S-**Noetherian Ring**) A Ring R is called an S-Noetherian ring if every ideal of R are S-finite.*

Example 4.1. *Let R be a Noetherian ring and $S \subseteq R$ be a multiplicatively closed subset. Then R is an S-Noetherian ring.*

Example 4.2. *Let S be a multiplicatively closed subset consisting of units of a ring R. Then R is S-Noetherian if and only if R is Noetherian.*

Example 4.3. *Let S be a multiplicatively closed subset of a ring R, and let I be an ideal of R. If $I \cap S \neq \varnothing$, then I is S-finite. For this, assume that $I \cap S \neq \varnothing$. Then there exists $s \in I \cap S$. Since $s \in I$, we have $sR \subseteq I$. Moreover, for any ideal I, we have $sI \subseteq sA$. The ideal $sR = (s)$*

is principal, hence finitely generated. Thus there exists a finitely generated ideal $sR \subset I$ such that $sI \subseteq sR$. Therefore, I is S-finite.

Example 4.4. *Let p be a prime number and consider the ring $R = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$. Then this ring R is not Noetherian. Indeed, consider the ideal*

$$I = \{(a_1, a_2, a_3, \dots) \in R \mid a_n = 0 \text{ for all but finitely many } n.\}$$

This ideal is not finitely generated; hence R is not Noetherian. For each $n \geq 1$, denote by $\overline{p^k}$ the residue class of p^kin $\mathbb{Z}/p^n\mathbb{Z}$, i.e.,$\overline{p^k} = p^k + p^n\mathbb{Z}$. Define elements

$$s_1 = (1, \overline{p}, \overline{p}, \overline{p}, \dots), s_2 = (1, \overline{p^2}, \overline{p^2}, \overline{p^2}, \dots), s_3 = (1, \overline{p^3}, \overline{p^3}, \overline{p^3}, \dots).$$

Let $S = (1, s_1, s_2, s_3)$. Then S is a multiplicative subset of R. and One can show that for every ideal J of R, there exists $s \in S$ such that sJ is becomes finitely generated . Hence R is S-Noetherian but not Notherian.

Example 4.5. *Let (p_n) be the sequence of prime numbers and define $R = \prod_{n=1}^{\infty} \mathbb{F}_{p_n}$, where $\mathbb{F}_{p_n}$ denotes the finite field with p_n elements. Then this ring R is not Noetherian. Because there exist an infinite strictly increasing chain of ideals: $\mathbb{F}_{p_1} \times 0 \times 0 \times \cdots \subsetneq \mathbb{F}_{p_1} \times \mathbb{F}_{p_2} \times 0 \times \cdots \subsetneq \cdots$ which does not stabilize. Hence R does not satisfy the ascending chain condition on ideals. However, choosing a suitable multiplicative set $S \subseteq R$, genrated by units and finite support elements, one can prove that R becomes S-Noetherian but not Notherian.*

4.2 Properties and characterizations

In this section, we provide basic properties and characterization of S-Noetherian ring as a generalization of corresponding properties and characterizations of Noetherian.

Proposition 4.1. *Let R be a ring and $S \subseteq R$ a multiplicative set. Then R is S-Noetherian if and only if each prime ideal of R disjoint with S is S-finite.*

Proof. Suppose that R is S-Noetherian. Then every ideal of R is S-finite. In particular, each prime ideal of R disjoint with S is S-finite.

Conversely, assume that every prime ideal of R disjoint with S is S-finite. We prove that R is S-Noetherian. Suppose, to the contrary, that there exists an ideal of R which is not S-finite. Let

$$\Sigma = \{\, I \subseteq R \mid I \text{ is an ideal of } R \text{ and } I \text{ is not } S\text{-finite} \,\}.$$

Then Σ is nonempty. By Zorn's Lemma, Σ contains a maximal element, say P. We claim that P is a prime ideal of R disjoint with S. Indeed, if $ab \in P$ and $a \notin P$, then the ideal $P + (a)$ properly contains P. By maximality of P, the ideal $P+(a)$ is S-finite, and from this it follows that P is S-finite, a contradiction. Thus P is prime. Moreover, if $P \cap S \neq \emptyset$, then $P = R$, which is impossible. Hence P is disjoint with S. By hypothesis, P is S-finite, contradicting the choice of $P \in \Sigma$. Therefore every ideal of R is S-finite. Hence R is S-Noetherian.

□

Proposition 4.2. *Let $R \subseteq B$ be a ring extension and let S be a multiplicative subset of R. Suppose that:*

1. *B is S-finite as an R-module;*
2. *For each prime ideal P of R such that $P \cap S = \emptyset$, the ideal PB is S-finite in B.*

Then R is S-Noetherian.

Proof. To prove that R is S-Noetherian, it suffices to show that every prime ideal of R disjoint from S is S-finite. Let P be a prime ideal of R such that $P \cap S = \emptyset$. By assumption, the extended ideal PB is S-finite in B. Hence there exist $s \in S$ and a finitely generated ideal $J \subseteq PB$ such that $s(PB) \subseteq J \subseteq PB$.Since B is S-finite as an R-module, there exist $t \in S$ and a finitely generated R-submodule M of B such that$tB \subseteq M$.Multiplying the inclusion $s(PB) \subseteq J$ by t, we obtain$ts(PB) \subseteq tJ$. Because M is finitely generated over R and $tB \subseteq M$, it follows that tJ is contained in a finitely generated R-submodule of B. Intersecting with R, we obtain $tsP \subseteq F \subseteq P$for some finitely generated ideal F of R.Thus P is S-finite. Since this holds for every prime ideal P disjoint from S, it follows that R is S-Noetherian. □

Proposition 4.3. *Let $A \subseteq B$ be an extension of rings and S a multiplicative set of R such that B is an S-finite R module. If B is an S-Noetherian ring, then R is S-Noetherian.*

Proof. To prove that R is S-Noetherian, It is sufficient to show that every prime ideal of R disjoint from S is S-finite. Let P be a prime ideal of R such that $P \cap S = \emptyset$. Since B is an S-Noetherian ring, every ideal of B is S-finite. In particular, the extended ideal PB of B is S-finite. Let P be a prime ideal of A such that $P \cap S = \emptyset$. Since B is an S-Noetherian ring, every ideal of B is S-finite. In particular, the extended ideal PB of B is S-finite. Hence, there exist $s \in S$ and a finitely generated ideal J of B such that$sPB \subseteq J \subseteq PB$. Since B is S-finite as an R-module, there exist $t \in S$ and a finitely generated R-submodule M of B such that$tB \subseteq M$. Multiplying the inclusion $sPB \subseteq J$ by t, we obtain$tsPB \subseteq tJ$. Because J is finitely generated as an ideal of B and $tB \subseteq M$, it follows that tJ is contained in a finitely generated R-submodule of B. Intersecting with R, we obtain a finitely generated ideal F of R such that $tsP \subseteq F \subseteq P$. Thus P is S-finite. Since this holds for every prime ideal P of R disjoint from S, it follows that R is S-Noetherian. □

Proposition 4.4. *Let $R \subseteq B$ be an extension of rings such that $IB \cap R = I$*

for each ideal I of R, and let $S \subseteq R$ be a multiplicative set. If B is an S-Noetherian ring, then R is S-Noetherian.

Proof. Let I be an ideal of R. Since B is S-Noetherian, the ideal IB of B is S-finite. Hence there exist $s \in S$ and elements $g_1, \ldots, g_n \in IB$ such that $sIB \subseteq (g_1, \ldots, g_n)B \subseteq IB$. Intersecting with R, we obtain $sIB \cap R \subseteq (g_1, \ldots, g_n)B \cap R \subseteq IB \cap R$. By the hypothesis $IB \cap R = I$, we have$sI = sIB \cap R \subseteq (g_1, \ldots, g_n)B \cap R \subseteq I$. Since each $g_i \in IB$, we can write $g_i = \sum_{j=1}^{m_i} a_{ij}b_{ij}$ with $a_{ij} \in I$ and $b_{ij} \in B$. Therefore,$(g_1, \ldots, g_n)B \cap R = (g_1, \ldots, g_n)R$, which is a finitely generated ideal of R. Hence there exists a finitely generated ideal $F = (g_1, \ldots, g_n)R$ of R such that $sI \subseteq F \subseteq I$. Thus I is S-finite. Since I was arbitrary, R is S-Noetherian. □

Example 4.6. *Let $R \subset B$ be an extension of rings and I an ideal of R. The equality $IB \cap R = I$ is in general false. Take the extension $R = \mathbb{Z} \subset B = \mathbb{Q}$ and $I = 2\mathbb{Z}$. Then $IB = 2\mathbb{Q} = \mathbb{Q}$, so $IB \cap R = \mathbb{Q} \cap \mathbb{Z} = \mathbb{Z} \neq 2\mathbb{Z} = I$.*

Theorem 4.1. *A ring R is Noetherian if and only if for each maximal ideal M of R, R is $(R \setminus M)$-Noetherian.*

Proof. Let I be an ideal of R. For each maximal ideal M of R, there exists $s_M \in R \setminus M$ and a finitely generated ideal F_M of R such that $s_M I \subset F_M \subset I$. If the ideal $J = (s_M; M \in \mathrm{Max}(R)) \neq R$, there exists $M_0 \in \mathrm{Max}(R)$ such that $J \subset M_0$, then $s_{M_0} \in J \subset M_0$, which is absurd. Then $J = R$ and there exist $M_1, \ldots, M_n \in \mathrm{Max}(R)$ such that $(s_{M_1}, \ldots, s_{M_n}) = A$. Then $I = RI = (s_{M_1}, \ldots, s_{M_n})I \subset F_{M_1} + \cdots + F_{M_n} \subset I$, so $I = F_{M_1} + \cdots + F_{M_n}$ is a finitely generated ideal. Then A is Noetherian. □

Theorem 4.2. *(**S-Version of Cohen's Theorem**) Let R be a commutative ring with identity and let $S \subseteq R$ be a multiplicatively closed subset. Then R is S-Noetherian if and only if every prime ideal of R disjoint from S is S-finite.*

Proof. Suppose that R is S-Noetherian.Then every ideal of R is S-finite. In particular, every prime ideal of R disjoint from S is S-finite.

Conversely, suppose that every prime ideal of R disjoint from S is S-finite. Assume, to the contrary, that R is not S-Noetherian. Then there exists an ideal of R which is not S-finite. Let$\Sigma = \{I \subseteq R \mid I$ is an ideal, $I \cap S = \varnothing$, I is not S-finite$\}$.Then Σ is nonempty. Partially order Σ by inclusion. By Zorn's lemma, Σ has a maximal element, say P. We claim that P is a prime ideal. Let $a, b \in R$ such that $ab \in P$ and $a \notin P$. Then $P \subsetneq P + (a)$. By maximality of P, the ideal $P + (a)$ is either S-finite or intersects S. Since $P \cap S = \varnothing$, it follows that $P + (a)$ must be S-finite. Similarly, if $b \notin P$, then $P + (b)$ is S-finite. Using the S-finiteness of $P + (a)$ and $P + (b)$, one obtains that P itself is S-finite, which contradicts the choice of P. Hence P is prime. Since $P \cap S = \varnothing$, by hypothesis P is S-finite, again a contradiction. Therefore Σ must be empty, and every ideal disjoint from S is S-finite. Hence R is S-Noetherian $\square$

Definition 4.3 ([7], p. 261, Definition 1)**.** *Let M be a module over a ring R. A prime ideal P is said to be associated with M if there exists $x \in M$ such that $P = \mathrm{Ann}(x)$. The set of all prime ideals associated with M is denoted by $\mathrm{Ass}_R(M)$, or simply $\mathrm{Ass}(M)$.*

Proposition 4.5 ([7],p.261, Definition 1)**.** *Let M be a module over a Noetherian ring R. Then the condition $M \neq \{0\}$ is equivalent to $\mathrm{Ass}_R(M) \neq \emptyset$.*

For modules over non-Noetherian rings however, the associated primes are not so useful and in fact do not exist for some modules M. For this consider the following examples.

Example 4.7. *[8] Let K be a field, $x_1, x_2, x_3, \ldots$ an infinite sequence of indeterminates, and denote by R the polynomial ring $K[x_1, x_2, x_3, \ldots]$. Then the A-module $A/(x_1^2, x_2^2, x_3^2, \ldots)$ does not possess an associated prime ideal since annihilator of any nonzero element of the module contains x_i^2 for all i but not x_j for some j.*

Definition 4.4. *[16] Let P be a prime ideal of a ring R, and M an R-module.*

1. *P is called a Zariski-Samuel prime of M if $\mathrm{Ann}(m)$ is P-primary for some $m \in M$. These primes will be denoted $Z - S_R(M)$.*

2. *If P is minimal over $\mathrm{Ann}(m)$ for some $m \in M$, then P is called a weak Bourbaki prime of M. We will denote the set of weak Bourbaki primes of M by $\mathrm{Ass}_f(M)$.*

3. *P is called a strong Krull prime of M if for every finitely generated ideal I contained in P there exists an $m \in M$ such that $I \subseteq \mathrm{Ann}(m) \subseteq P$. We will let $sK_R(M)$ denote the set of strong Krull primes of M.*

4. *P is called a Krull prime of M if for each $p \in P$ there exists $m \in M$ such that $p \in \mathrm{Ann}(m) \subseteq P$. We will denote the set of these primes by $K_R(M)$.*

Proposition 4.6. *[16] For an R-module M, we have*

$$\mathrm{Ass}(M) \subseteq Z - S(M) \subseteq \mathrm{Ass}_f(M) \subseteq sK(M) \subseteq K(M).$$

Proposition 4.7. *[1] If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. Then M is S-Noetherian if and only if so are M' and M''. In particular, if R is S-Noetherian, then so is every finitely generated R-module.*

Let R be a ring and S a multiplicatively closed subset of R. We say that a prime ideal P of R is S-proper if $P \cap S = \varnothing$. The set of all S-proper prime ideals of R is denoted by $\mathrm{Spec}^S(R)$. Also, the set of all nilpotent elements of R is denoted by $\mathrm{Nil}(R)$. We prove that S-Artinian ring is S-Noetherian. For this, first we prove the following two lemmas.

Lemma 4.1. *Let R be a ring and S a multiplicatively closed subset of R which contains no zero divisors of R. Then* $\mathrm{Nil}(R) = \bigcap_{P \in \mathrm{Spec}^S(R)} P$.

Proof. Since intersection of all prime ideals is contained in the intersection of all prime ideals that are disjoint with S, $\mathrm{Nil}(R) = \bigcap_{P \in \mathrm{Spec}(R)} P \subseteq \bigcap_{P \in \mathrm{Spec}^S(R)} P$. Now, let $a \in \bigcap_{P \in \mathrm{Spec}^S(R)} P$ and consider the set $T = \{a^i b : i \in \mathbb{N} \cup \{0\}, b \in S\}$. Then clearly T is a multiplicatively closed subset of R containing S. Let $P \in \mathrm{Spec}(R)$. If $P \in \mathrm{Spec}^S(R)$, then $a \in P \cap T$, and so $P \cap T \neq \varnothing$. Also if $P \notin \mathrm{Spec}^S(R)$, then $P \cap S \neq \varnothing$, and so $P \cap T \neq \varnothing$ since $S \subseteq T$. This implies that $T^{-1}R$ has no proper prime ideal, and so $T^{-1}R = 0$. Consequently, $0 \in T$ which implies that $a^k b = 0$ for some $k \in \mathbb{N}$ and $b \in S$. Since S contains no zero-divisors of R, we have $a^k = 0$. This implies that $a \in \mathrm{Nil}(R)$, and hence $\bigcap_{P \in \mathrm{Spec}^S(R)} P \subseteq \mathrm{Nil}(R)$, as desired. □

Lemma 4.2. *Let R be a ring and S a multiplicatively closed subset of R which contains no zero divisors of R. If R is an S-Artinian ring, then nilradical* $\mathrm{Nil}(R)$ *of R is nilpotent.*

Proof. Let $N = \mathrm{Nil}(R)$ be the nilradical of R. Consider the descending chain $N \supseteq N^2 \supseteq \cdots \supseteq N^n \supseteq \cdots$ in R. Since R is S-Artinian, there exist $s \in S$ and $k \geq 1$ such that $sN^i \subseteq N^k$ for all $i \geq k$. Put $I = N^k$, then $sI \subseteq N^i$ for all $i \geq k$. We need to show $I = 0$.

On contrary, suppose $I \neq 0$. Let $\mathcal{X}$ be the set of all ideals J of R such that $JI \neq 0$. Since $I \neq 0$ and $s \in S$ is a non-zero-divisor of R, $sI \neq 0$ and $sI \subseteq I^2$, whence $I \in \mathcal{X}$. Therefore, $\mathcal{X}$ is non-empty and by [[25], Theorem 2.1] , $\mathcal{X}$ has an S-minimal element, say K. Obviously $I \nsubseteq 0$, and so there exists $r \in K$ such that $rI \neq 0$. Also $rI \subseteq RI \subseteq K$ and K is S-minimal, so there exists $t \in S$ such that $tK \subseteq rI$. Write $tr = xy$ for some $y \in I \subseteq N$ and multiplying repeatedly with t, we get $t^i r = xy^i$ for all $i \geq 1$. Since $y \in N$, $y^n = 0$ for some $n \in \mathbb{N}$, so $t^n r = xy^n = 0$ which implies that $r = 0$ as S contains no zero divisors of R. Thus, we get a contradiction as $rI \neq 0$. Hence, $I = 0$, as required. □

Now we prove S-version of a well known result which states that an Artinian ring is Noetherian [[**?**],Theorem 3.2]

Theorem 4.3. *Let R be a ring and S a multiplicatively closed subset of R which contains no zero-divisors of R. If R is S-Artinian, then R is an S-Noetherian ring.*

Proof. Since R is S-Artinian, so by [[25],Theorem 2.4], there are only finitely many S-proper prime ideals. Also by [[25],Proposition 2.5], every S-proper prime ideal is maximal. Let $m_1, m_2, \ldots, m_r$ be all distinct S-proper maximal ideals of R, then by Lemma 4.1, $\mathrm{Nil}(R) = m_1 \cap m_2 \cap \cdots \cap m_r$. Also, by Lemma 4.2, $\mathrm{Nil}(R)^k = 0$ for some $k \geq 1$. Thus, $\prod_{i=1}^{r} m_i^k \subseteq (m_1 \cap \cdots \cap m_r)^k = 0$, and so $m_1 m_2 \cdots m_r = 0$. Now consider the chain of ideals $m_0 = R \supseteq m_1 \supseteq m_1 m_2 \supseteq \cdots \supseteq m_1 m_2 \cdots m_r = 0$. Since each factor $m_0 \cdots m_{i-1}/m_0 \cdots m_i$ is a vector space over the field R/m_i for $i = 1, 2, \ldots, r$. Hence, each factor is Noetherian, and so S-Noetherian. Thus, by Proposition 4.8, R is S-Noetherian. □

Recall that if M is a Noetherian R-module, then $R/\operatorname{Ann}(M)$ is a Noetherian ring (see [[22],Theorem 3.5]). Now we extend this result to the case of S-Noetherian modules.

Theorem 4.4. *Let R be a ring, M be an R-module and S a multiplicatively closed subset of R which contains no zero divisors of M. If an R-module M is S-Noetherian, then $R/\mathrm{Ann}(M)$ is an $\overline{S}$-Noetherian ring.*

Proof. Since M is an S-Noetherian R-module, M is $\pi(S)$-Noetherian $\overline{R}$-module, where $\pi : R \to \overline{R}$ is canonical homomorphism and $\overline{R} = R/\mathrm{Ann}(M)$. This implies that there exist $\overline{t} \in \pi(S)$, where $t \in S$ and a finitely generated R-submodule F of M such that $tM \subseteq F$. Put $F = R\overline{x}_1 + R\overline{x}_2 + \cdots + R\overline{x}_n$ for some $x_1, x_2, \ldots, x_n \in M$ and $M^{(n)} = M \oplus M \oplus \cdots \oplus M$. Then by Proposition 4.8, $M^{(n)}$ is a $\pi(S)$-Noetherian $\overline{R}$-module. Since $tM \subseteq F$, then $\overline{t}M^{(n)} \subseteq F^{(n)} \subseteq M^{(n)}$. Now define $\phi : R \to M^{(n)}$ by $\phi(\overline{a}) =$

$(\overline{tax}_1, \overline{tax}_2, \ldots, \overline{tax}_n)$. Then ϕ is an $\overline{R}$-module homomorphism. Let $\overline{a} \in \ker \phi$, then $\overline{tax}_i = 0$ for all $i = 1, 2, \ldots, n$. Consequently, $\overline{taM} \subseteq \overline{taF} = 0$ which implies that $\overline{t^2aM} = 0$. This implies that $(t^2a + \mathrm{Ann}(M))M = 0$, and so $t^2aM = 0$. Consequently $t^2a = 0$ as $t^2 \in S$ and S contains no zero-divisors of M. Thus, $a \in \mathrm{Ann}(M)$, and so $\overline{a} = 0$. This implies that $\overline{R}$ is a submodule of $M^{(n)}$. Hence, $\overline{R}$ is an S-Noetherian ring. □

Corollary 4.1. *Let R be a ring and M an S-finite faithful R-module. Then M is an S-Noetherian R-module if and only if R is an S-Noetherian ring.*

Now we prove the existence of associated prime ideals to a module over S-Noetherian ring.

Proposition 4.8. *Let R be a ring, M an R-module and S a multiplicatively closed subset of R containing no zero-divisors of M. If R is an S-Noetherian ring, then $\mathrm{Ass}_(M) \neq \emptyset$ if and only if $M \neq 0$.*

Proof. If $M = 0$, then obviously $\mathrm{Ass}_R(M) = \emptyset$. Assume that $M \neq 0$. Consider the set

$X = \{\mathrm{Ann}(y) : 0 \neq y \in M\}$.

Clearly $X \neq \emptyset$. Since R is S-Noetherian, by [[6], Theorem 2.3], there exists an S-maximal element, say $P \in X$. Then $P = \mathrm{Ann}(x)$ for some $0 \neq x \in M$. We show that P is a prime ideal.

For this, let $a, b \in R$ such that $ab \in P$ and $b \notin P$. Then $abx = 0$ but $bx \neq 0$, and so $a \in \mathrm{Ann}(bx)$. Since $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(bx)$ and $\mathrm{Ann}(bx) \in X$, there exists $s \in S$ such that $s\mathrm{Ann}(bx) \subseteq \mathrm{Ann}(x)$ as P is S-maximal element of X. Let $\alpha \in \mathrm{Ann}(bx)$, then $s\alpha x = 0$ implies that $\alpha x = 0$ as s is not a zero-divisor of M. Thus, $\alpha \in \mathrm{Ann}(x) = P$, and so $\mathrm{Ann}(bx) = \mathrm{Ann}(x)$ which implies that $a \in \mathrm{Ann}(x) = P$. Therefore, P is a prime ideal. Hence, $\mathrm{Ass}_R(M) \neq \emptyset$. □

It is well known that the notion of associated prime ideals, strong Krull prime ideals, weak Bourbaki prime ideals, and Zariski-Samuel prime ideals coincide for modules over Noetherian rings. We extend this result for the case of modules over S-Noetherian rings.

Proposition 4.9. *Let R be an S-Noetherian ring and M an R-module. Then $\mathrm{Ass}_R(M) = Z_S R(M) = \mathrm{Ass}_f(M) = SK_R(M)$.*

Proof. By Proposition 4.7, $\mathrm{Ass}_R(M) \subseteq Z{-}SR(M) \subseteq \mathrm{Ass}_f(M) \subseteq SK_R(M)$. So it is enough to show that $\mathrm{Ass}_R(M) = SK_R(M)$. For this, let $P \in SK_R(M)$, then P is S-finite as R is S-Noetherian. So there exist a finitely generated ideal I of R and $s \in S$ such that $sP \subseteq I \subseteq P$. Then by definition of strong Krull prime ideal, there exists a nonzero $x \in M$ such that $I \subseteq \mathrm{Ann}(x) \subseteq P$, and so $sP \subseteq \mathrm{Ann}(x)$. Now let $\alpha \in P$, then $s\alpha \in \mathrm{Ann}(x)$ and so $s\alpha x = 0$. This implies that $\alpha x = 0$ and so $\alpha \in \mathrm{Ann}(x)$ which implies $P \subseteq \mathrm{Ann}(x)$. Thus, $P = \mathrm{Ann}(x)$ and so $P \in \mathrm{Ass}_R(M)$, as desired.

Let R be a ring, S a multiplicatively closed subset of R, and M an R-module. Recall by [2] that a descending chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of submodules of M, and denoted by (M_n). For an ideal I of R, a filtration (M_n) is said to be I-filtration if $IM_n \subseteq M_{n+1}$ for all n and (M_n) is said to be stable I-filtration if $IM_n = M_{n+1}$ for all sufficiently large n. Motivated by this, we define S-stable I-filtration as a generalization of stable I-filtration. This generalization allows us to prove the Krull's intersection theorem for S-Noetherian local ring. An I-filtration (M_n) is said to be S-stable if there exists an $s \in S$ such that $sM_{n+1} \subseteq IM_n$ for sufficiently large n. □

Now we provide an example of an S-stable I-filtration which is not a stable I-filtration.

Example 4.8. *Consider $M = \mathbb{Z}$ as $\mathbb{Z}$-module, $I = 6\mathbb{Z}$ and multiplicatively closed set $S = \{3^n : n \in \mathbb{N} \cup \{0\}\}$. Let $M_n = 2^n\mathbb{Z}$, then (M_n) is an I-filtration of M. Put $s = 3$. Then $sM_{n+1} \subseteq IM_n$ for all $n \in \mathbb{N} \cup \{0\}$, and so*

(M_n) is an S-stable I-filtration. However, (M_n) is not a stable I-filtration since $IM_n = 3M_{n+1} \neq M_{n+1}$.

Let R be a ring and I be an ideal of R. Then $R^* = \bigoplus_{n\geq 0} I^n$ is a graded ring. Similarly, if M is an R-module and (M_n) is an I-filtration of M, then $M^* = \bigoplus_{n\geq 0} M_n$ is a graded R^*-module since $I^m M_n \subseteq M_{m+n}$.

In order to prove Krull's intersection theorem for S-Noetherian local ring, we need S-version of Artin-Rees Lemma. Now, we obtain S-version of Artin-Rees lemma. For this we need the following two lemmas.

Lemma 4.3. *Let R be an S-Noetherian ring and I be an ideal of R. Then $R^* = \bigoplus_{n\geq 0} I^n$ is an S-Noetherian ring.*

Proof. Since R is S-Noetherian, I is S-finite. This implies that there exist $s \in S$ and a finitely generated ideal $F = (x_1, x_2, \ldots, x_n)$ of R such that $sI \subseteq F \subseteq I$. Consequently, $sR^* \subseteq R[x_1, x_2, \ldots, x_n]$. This implies R^* is an S-finite R-algebra. Therefore, by [18, Lemma 6], R^* is an S-Noetherian ring. □

Lemma 4.4. *Let R be an S-Noetherian ring, I an ideal of R, M an S-finite R-module and (M_n) an I-filtration of M. Then the following statements are equivalent:*

1. *M^* is an S-finite R^*-module.*
2. *The filtration (M_n) is S-stable.*

Proof. Since R is an S-Noetherian ring and M is S-finite, by Proposition 4.8, M is an S-Noetherian R-module. Therefore, each M_n is S-finite, and so each $Q_n = \bigoplus_{r=0}^{n} M_r$ is S-finite as an R-module. Notice that Q_n is a subgroup of M^* but not an R^*-submodule in general. Consider the R^*-submodule $M_n^* = M_0 \oplus \cdots \oplus M_n$ of M^* generated by Q_n. Since Q_n is an S-finite R-module, M_n^* is an S-finite R^*-module. Clearly $M_0^* \subseteq M_1^* \subseteq M_2^* \subseteq$

$\cdots \subseteq M_n^* \subseteq \cdots$ is an increasing chain in M^* whose union is M^*. Since R^* is S-Noetherian by Lemma 4.3, M^* is an S-finite as an R^*-module if and only if chain (M_n^*) is S-stationary, i.e., there exists $s \in S$ and $n \geq 1$ such that $sM_k^* \subseteq M_n^*$ for all $n \geq k$, if and only if $sM_{k+r} \subseteq M_k \quad$ for all $r \geq 0$, if and only if the filtration (M_n) is S-stable, as desired. □

In the next proposition, we obtain an S-version of Artin–Rees lemma for S-Noetherian ring.

Proposition 4.10. *Let R be an S-Noetherian ring, I an ideal of R, M an S-finite R-module and (M_n) an S-stable I-filtration. If N is a submodule of M, then $(N \cap M_n)$ is an S-stable I-filtration of N.*

Proof. As $I(N \cap M_n) \subseteq IN \cap IM_n \subseteq N \cap M_{n+1}$, so $(N \cap M_n)$ is an I-filtration of N. This implies that $N^* = \bigoplus_{n\geq 0}(N \cap M_n)$ is a graded R^*-submodule of M^*. By Lemma 4.3, R^* is S-Noetherian. Also by Lemma 4.4, M^* is S-finite R^*-module. Consequently, by Proposition 4.8, M^* is S-Noetherian as an R^*-module. This implies that N^* is S-finite as R^*-module. Then result is followed from Lemma 4.4. □

Now we are in a position to prove the Krull's intersection theorem for S-Noetherian local ring.

Theorem 4.5. *Let R be an S-Noetherian local ring and M an S-finite R-module, where S is a multiplicatively closed subset of R containing no zero-divisors of M. If I is a proper ideal of R, then $\bigcap_{n\geq 0} I^n M = 0$.*

Proof. Let $M_n = I^n M$ and $N = \bigcap_{n\geq 0} M_n$. Then $N = N \cap M_n$ for all $n \geq 0$. By Proposition 4.11, there exist $s \in S$ and $n \geq 1$ such that $s(N \cap M_{n+1}) \subseteq I(N \cap M_n)$, i.e., $sN \subseteq IN$. By Proposition 4.8, M is S-Noetherian, and so N is S-finite. Then by [[15],Theorem 2.1], there exist $s \in S$ and $a \in I$ such that $(sa)N = 0$, which implies that $tN = 0$. Since S contains no zero-divisors of M, $IN = 0$. Hence, $\bigcap_{n\geq 0} I^n M = 0$. □

Corollary 4.2. *Let R be an S-Noetherian local domain, and I a proper ideal of R. Then $\bigcap_{n\geq 0} I^n = 0$.*

4.3 Polynomial ring over S- Noetherian ring

Definition 4.5 ([5],Page13, Definition 4.1)**.** *Let R be a ring and $S \subseteq R$ a multiplicative set. We say that S is anti-Archimedean if for each $s \in S$,*

$$\bigcap_{n:1}^{\infty}(s^n R) \cap S \neq \emptyset.$$

Example 4.9. *If $S \subseteq U(R)$, then S is anti-Archimedean.*

Example 4.10. *(Hamed and Hizem 2016) [5]Every finite multiplicative set S of a ring R is anti-Archimedean. Indeed, put $S = \{s_1, \ldots, s_k\}$ and $t = s_1 \cdots s_k \in S$. Let $s \in S$. For each integer $n \geq 1$, $s^n \in S$, and then $s^n = s_i$, with $i \in \{1, \ldots, k\}$. Then $t \in s_i R = s^n R$, so $t \in \bigcap_{n:1}^{\infty}(s^n R) \cap S$. For example, $S = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\}$ is an anti-Archimedean multiplicative set of the ring $\mathbb{Z}/12\mathbb{Z}$.*

Note that a finite multiplicative set $S = \{s_1, \ldots, s_k\}$ not containing zero divisors is included in $U(R)$. Indeed, since $s_1 \cdots s_k \in S$, there exists $j \in \{1, \ldots, k\}$ such that $s_1 \cdots s_k = s_j$, so $s_1 \cdots s_{j-1} s_{j+1} \cdots s_k = 1$. Then

$$s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_k \in U(R)$$

. On the other hand, $s_j^2 \in S$, then $s_j^2 = s_i$ with $i \in \{1, \ldots, k\}$. If $i \neq j$, then $s_i \in U(R)$, so $s_j \in U(R)$. If $i = j$, then $s_j^2 = s_j$, so $s_j = 1 \in U(R)$.

Example 4.11. *[5] Let R be an integral domain with quotient field K and complete integral closure equal to K. Then $R \setminus (0)$ is an anti-Archimedean multiplicative set. Indeed, let $0 \neq a \in R$. There exists $0 \neq s \in R$ such that $\frac{s}{a^n} \in R$ for each integer $n \geq 1$, so $s \in a^n R$. Then*

$$s \in \bigcap_{n:1}^{\infty} a^n R.$$

Example 4.12. *[5] Let R be a valuation domain which is not a field and without a height one prime ideal. The multiplicative set $R \setminus (0)$ is anti-Archimedean. Indeed, let $0 \neq a \in R$. Suppose that a is in all the nonzero*

prime ideals of R. The intersection P of all these ideals is prime because they are comparable, and we have $(0) \subset (a) \subseteq P$. Then the height of P is 1, which is absurd. There exists a nonzero prime ideal P_0 of R strictly included in (a). Since R is a valuation domain, for each integer $n \geq 1$, $P_0 \subseteq a^n R$. Then $(0) \neq P_0 \subseteq \bigcap_{n:1}^{\infty} a^n R$.

Example 4.13. *(Park et al. 2019) [5] We will generalize the preceding example. Let R be a Prüfer domain without a height one prime ideal. Suppose that each principal ideal of R has only finitely many minimal prime ideals. Then $R \setminus (0)$ is an anti-Archimedean multiplicative set. Indeed, let $0 \neq a \in R$ and $P_1, \ldots, P_m$ the minimal prime ideals over (a). Since R has no height one prime ideal, for each i, there is a nonzero prime ideal Q_i of A such that $Q_i \subset P_i$. The minimality of P_i over (a) implies that $a \notin Q_i$. So $a^n \notin Q_i$ for each $n \in \mathbb{N}^*$. Let $M \in \mathrm{Max}(R)$ and $n \in \mathbb{N}^*$.*

First case: There is i such that $Q_i \subseteq M$. In this case, $Q_i R_M$ is a prime ideal of R_M. Since $R \cap Q_i R_M = Q_i$, then $a^n \notin Q_i R_M$. Since R_M is a valuation domain, $Q_i R_M \subseteq a^n R_M$. So $aQ_1 \cdots Q_m R_M \subseteq a^n R_M$.

Second case: For each i, $Q_i \nsubseteq M$. In this case, $a \notin M$ because, in the contrary case, M must contain at least one P_i, and then it contains Q_i, which is absurd. We have $Q_1 \cdots Q_m \nsubseteq M$. So $Q_1 \cdots Q_m A_M = R_M = a^n R_M$.

We conclude that for each $M \in \mathrm{Max}(R)$ and $n \in \mathbb{N}^*$, $Q_1 \cdots Q_m R_M \subseteq a^n R_M$. Thus $(0) \neq Q_1 \cdots Q_m \subseteq \bigcap_{n:1}^{\infty} a^n R$.

Example 4.14. *[5] For each ring R, the multiplicative set $\{X^n; n \in \mathbb{N}\}$ of $R[X]$ and $R[[X]]$ is not anti-Archimedean because $\bigcap_{n:1}^{\infty}(X^n) = (0)$.*

Theorem 4.6 ([5],Page14, Theorem 4.2)**.** *Let R be a ring and S an anti-Archimedean multiplicative set of R. If R is S-Noetherian, then it is the same for the polynomial ring $R[X_1, \ldots, X_n]$.*

Proof. Since S is anti-Archimedean in any ring containing R, we may suppose $n = 1$. Put $X_1 = X$. Let I be a nonzero ideal of $R[X]$ and J the ideal of R formed by 0 and the leading coefficients of the nonzero polynomials

in I. Since R is S-Noetherian, there exist $s \in S$ and $a_1, \ldots, a_n \in J \setminus \{0\}$ such that $sJ \subseteq (a_1, \ldots, a_n)R$. Let $f_1, \ldots, f_n \in I$ be such that a_i is the leading coefficient of f_i, $1 \leq i \leq n$. By multiplying the f_i with convenient powers of X, we may suppose that $\deg f_1 = \cdots = \deg f_n = d$. Consider the finitely generated R-module $M = R + RX + RX^2 + \cdots + RX^{d-1}$. By Theorem 3.5, M is a S-Noetherian R-module. Its sub-module $M \cap I = T$ is S-finite. There exist $t \in S$ and $g_1, \ldots, g_m \in T$ such that $tT \subseteq (g_1, \ldots, g_m)A$. Let $f \in I$. By repeated subtractions of linear combinations of $f_1, \ldots, f_n$ with coefficients in $R[X]$, we can find $q \in \mathbb{N}^*$ such that $s^q f \in (f_1, \ldots, f_n)R[X] + T$. Indeed, $f = b_0 + b_1 X + \cdots + b_k X^k$, $k \geq d$, $sb = \alpha_1 a_1 + \cdots + \alpha_n a_n$ with $\alpha_i \in R$. Then $sf - \sum_{i=1}^n \alpha_i X^{k-d} f_i \in I$ with degree $k' \leq k-1$. We restart, $s\left(sf - \sum_{i=1}^n \alpha_i X^{k-d} f_i\right) - \sum_{i=1}^n \beta_i X^{k'-d} f_i \in I$ of degree $k'' \leq k' - 1$ with $\beta_i \in R$. We continue until we arrive at a polynomial of degree $\leq d-1$. Let $w \in \bigcap_{j:1}^{\infty}(s^j R) \cap S$. Then $w = s^q c$ with $c \in R$. Then $twf = ts^q cf \in (f_1, \ldots, f_n)A[X] + tA[X] \subseteq (f_1, \ldots, f_n)R[X] + (g_1, \ldots, g_m)R[X] = (f_1, \ldots, f_n, g_1, \ldots, g_m)R[X]$. Then

$$(tw)I \subseteq (f_1, \ldots, f_n, g_1, \ldots, g_m)R[X]$$

with $f_1, \ldots, f_n, g_1, \ldots, g_m \in I$ and $tw \in S$ □

Example 4.15. *We retrieve the Hilbert basis theorem. If R is a Noetherian ring, then it is the polynomial ring $R[X_1, \ldots, X_n]$.*

Example 4.16. *For any multiplicative set S of a ring R, the ring $B = R[X_n; n \in \mathbb{N}]$ is not S-Noetherian. Indeed, suppose that the ideal $I = (X_n; n \in \mathbb{N})$ is S-finite. There exist $s \in S$ and $k \in \mathbb{N}$ such that $sI \subseteq (X_0, \ldots, X_k)$. There exist $f_0, \ldots, f_k \in B$ such that $sX_{k+1} = X_0 f_0 + \cdots + X_k f_k$. Putting $X_0 = \cdots = X_k = 0$, we find $sX_{k+1} = 0$, and then $s = 0$, which is absurd*

4.4 Formal power series over S-Noetherian ring

Theorem 4.7 ([5],Page20, Theorem 5.1)**.** *Let R be a ring and S an anti-Archimedean multiplicative set of R not containing zero divisors. If R is S-Noetherian, it is the same for the formal power series ring $R[[X_1, \ldots, X_n]]$.*

Proof. Since S is anti-Archimedean in any ring containing R and it does not contain zero divisors in $R[[X_1, \ldots, X_n]]$, we may suppose $n = 1$. Put $X_1 = X$. By Corollary 2.2 [5], it suffices to show that each prime ideal P of $R[[X]]$ is S-finite. Consider the ideal $P' = \{f(0);\ f \in P\}$ of R. There exist $s \in S$ and $f_1, \ldots, f_n \in P$ such that $sP' \subseteq (f_1(0), \ldots, f_n(0))$.

First case: $X \in P$. Then $P = P' + XR[[X]]$. We will show that $sP \subseteq (f_1, \ldots, f_n, X) \subseteq P$. Put $f_i = f_i(0) + Xh_i$ with $h_i \in R[[X]]$, $1 \leq i \leq n$. Let $f \in P$, $f = f(0) + Xh$ with $h \in R[[X]]$. Then $sf = sf(0) + Xsh = \sum_{i=1}^{n} a_i f_i(0) + Xsh$ with $a_1, \ldots, a_n \in R$. Then $sf = \sum_{i=1}^{n} a_i(f_i - Xh_i) + Xsh = \sum_{i=1}^{n} a_i f_i + X\left(sh - \sum_{i=1}^{n} a_i h_i\right) \in (f_1, \ldots, f_n, X)$.

Second case: $X \notin P$. Let $t \in (\bigcap_{i:1}^{\infty} s^i R) \cap S$. We will show that $tP \subseteq (f_1, \ldots, f_n) \subseteq P$. Put $a_i = f_i(0)$, $1 \leq i \leq n$. Let $f \in P$. Then $sf(0) = \sum_{i=1}^{n} b_{i0} a_i$ with $b_{i0} \in R$, then $sf - \sum_{i=1}^{n} b_{i0} f_i = Xg_1 \in P$. Then $g_1 \in P$ and $sf = \sum_{i=1}^{n} b_{i0} f_i + Xg_1$. In the same way, $sg_1 = \sum_{i=1}^{n} b_{i1} f_i + Xg_2$ with $b_{i1} \in R$ and $g_2 \in P$. Then

$$s^2 f = \sum_{i=1}^{n} sb_{i0} f_i + Xsg_1 = \sum_{i=1}^{n} sb_{i0} f_i + X\left(\sum_{i=1}^{n} b_{i1} f_i + Xg_2\right)$$

$$= \sum_{i=1}^{n} (sb_{i0} + b_{i1}X) f_i + X^2 g_2.$$

In the same way, $sg_2 = \sum_{i=1}^{n} b_{i2} f_i + Xg_3$ with $b_{i2} \in R$ and $g_3 \in P$. Then

$$s^3 f = \sum_{i=1}^{n} (s^2 b_{i0} + sb_{i1}X) f_i + X^2 sg_2 = \sum_{i=1}^{n} (s^2 b_{i0} + sb_{i1}X) f_i + X^2 \left(\sum_{i=1}^{n} b_{i2} f_i + Xg_3\right)$$

$$= \sum_{i=1}^{n} (s^2 b_{i0} + s b_{i1} X + b_{i2} X^2) f_i + X^3 g_3.$$

By induction, for each $k \in \mathbb{N}$,

$$s^{k+1} f = \sum_{i=1}^{n} \left(s^k b_{i0} + s^{k-1} b_{i1} X + \cdots + s b_{ik-1} X^{k-1} + b_{ik} X^k \right) f_i + X^{k+1} g_{k+1}$$

with $g_{k+1} \in R[[X]]$.

Since S does not contain zero divisor, $R \subseteq S^{-1}R = B$ and in $B[[X]]$, $f = \sum_{i=1}^{n} \left(\frac{b_{i0}}{s} + \frac{b_{i1}}{s^2} X + \cdots + \frac{b_{ik-1}}{s^k} X^{k-1} + \frac{b_{ik}}{s^{k+1}} X^k \right) f_i + \frac{X^{k+1}}{s^{k+1}} g_{k+1}$. Put $h_i = \sum_{j=0}^{\infty} \frac{b_{ij}}{s^{j+1}} X^j \in B[[X]], \quad 1 \leq i \leq n$. Put $t = s^j c_j$ with $c_j \in R$ for $j \in \mathbb{N}$. Then $h_i = \sum_{j=0}^{\infty} \frac{b_{ij} c_{j+1}}{t} X^j, \quad$ so $\quad t h_i = \sum_{j=0}^{\infty} b_{ij} c_{j+1} X^j \in R[[X]]$. We will show that $f = h_1 f_1 + \cdots + h_n f_n$, and then $tf = (th_1) f_1 + \cdots + (th_n) f_n \in (f_1, \ldots, f_n) R[[X]]$. For each $g = \sum_{i=0}^{\infty} x_i X^i \in B[[X]]$ and $k \in \mathbb{N}$, we note $g^{(k)} = \sum_{i=0}^{k} x_i X^i \in B[X]$ a polynomial of degree $\leq k$ and $g = g^{(k)} + X^{k+1} h$ with $h \in B[[X]]$. We have

$$f = f^{(k)} + X^{k+1} u = \sum_{i=1}^{n} h_i^{(k)} \left(f_i^{(k)} + X^{k+1} r_i \right) + \frac{X^{k+1}}{s^{k+1}} g_{k+1},$$

with $u, r_i \in B[[X]]$. Then $f^{(k)} - \sum_{i=1}^{n} h_i^{(k)} f_i^{(k)} = X^{k+1} v$, with $v \in B[[X]]$. The order $\omega\left(f^{(k)} - \sum_{i=1}^{n} h_i^{(k)} f_i^{(k)} \right) \geq k+1$. For each $k \in \mathbb{N}$, the coefficient of X^k in $f - \sum_{i=1}^{n} h_i f_i$ is equal to that of X^k in $f^{(k)} - \sum_{i=1}^{n} h_i^{(k)} f_i^{(k)}$, which is zero. Then $f - \sum_{i=1}^{n} h_i f_i = 0$. □

Corollary 4.3 ([5],5.2, I. Kaplansky)**.** *If R is a Noetherian ring, the formal power series ring $R[[X_1, \ldots, X_n]]$ is also Noetherian.*

Proof. Take $S = \{1\}$. □

4.5 S-Primary decomposition

In this section, we discuss the concept of S-primary decomposition as a generalization of primary decomposition. We say that an ideal (disjoint

from S) of a ring R is S-primary decomposable or has an S-primary decomposition if it can be written as a finite intersection of S-primary ideals of R. First we provide an example of an S-Noetherian ring in which primary decomposition does not exist (see Example 4.18) which asserts that an S-Noetherian ring need not be a Laskerian ring in general.

Example 4.17. *Let $R = F[x_1, x_2, \ldots, x_n, \ldots]$ be the polynomial ring in infinitely many indeterminates over a field F. Since R has an ascending chain of prime ideals $(x_1) \subseteq (x_1, x_2) \subseteq \cdots \subseteq (x_1, x_2, \ldots, x_n) \subseteq \cdots$ which does not terminate, R has no Noetherian spectrum. This implies that R is a non-Laskerian ring, by Theorem 4 of [9]. Consider the multiplicative set $S = R \setminus \{0\}$. Then by Proposition 2(a) of [1], R is an S-Noetherian ring. Hence, R is an S-Noetherian ring but not Laskerian.*

Recall that $f : R \to S^{-1}R$ denotes the usual homomorphism of rings given by $f(r) = r/1$. For any ideal I of R, the $f^{-1}(S^{-1}I)$ is called the contraction of I with respect to S, that is, $\{a \in R : a/1 \in S^{-1}I\}$ is denoted by $S(I)$. Notice that $I \subseteq S(I)$. Thus, we need an S-version of primary decomposition of ideals. Now we define the concept of S-primary decomposition of ideals as a generalization of primary decomposition.

Definition 4.6. *Let R be a ring and let S be a multiplicative set of R. Let I be an ideal of R such that $I \cap S = \varnothing$. We say that I admits an S-primary decomposition if I is a finite intersection of S-primary ideals of R. In such a case, we say that I is* S-decomposable.

An S-primary decomposition $I = \bigcap_{i=1}^{n} Q_i$ of I with $\operatorname{rad}(Q_i) = P_i$ *for each $i \in \{1, 2, \ldots, n\}$ is said to be* minimal *if the following conditions hold:*

1. *$S(P_i) \neq S(P_j)$ for all distinct $i, j \in \{1, 2, \ldots, n\}$;*
2. *$S(Q_i) \not\supseteq \bigcap_{\substack{j \in \{1,2,\ldots,n\} \\ j \neq i}} S(Q_j)$ for each $i \in \{1, 2, \ldots, n\}$ (equivalently, $S(Q_i) \not\supseteq \bigcap_{\substack{j \in \{1,2,\ldots,n\} \\ j \neq i}} Q_j$ for each $i \in \{1, 2, \ldots, n\}$).*

From Definition 4.6, the concepts of S-primary decomposition and primary decomposition coincide for $S = \{1\}$. The following example shows

that the concept of S-primary decomposition is a proper generalization of the concept of primary decomposition.

Example 4.18. *Consider the Boolean ring $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ (countably infinite copies of $\mathbb{Z}_2$). According to Theorem 1 of [27], the zero ideal $(0) = (0, 0, 0, \ldots)$ in R does not have the primary decomposition. Clearly, R is not Noetherian. Consider the multiplicative set $S = \{1_R = (1, 1, 1, \ldots), s = (1, 0, 0, \ldots)\}$. Let I be an ideal of R. Either $s \in I$ or $s \notin I$. If $s \in I$, then $sI \subseteq Rs \subseteq I$ and Rs is a finitely generated ideal of R, it follows that I is S-finite. If $s \notin I$, then $I \subseteq (0) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$. In such a case, $sI \subseteq (0) \times (0) \times (0) \times \cdots \subseteq I$. Hence, I is S-finite. This shows that R is an S-Noetherian ring. Next, we show that (0) is an S-primary ideal of R. First, we observe that $(0) \cap S = \varnothing$. Now, let $a = (a_n)_{n\in\mathbb{N}}$, $b = (b_n)_{n\in\mathbb{N}} \in R$ such that $ab = 0$, where each $a_i, b_i \in \mathbb{Z}_2$. This implies that $a_n b_n = 0$ for all $n \in \mathbb{N}$, in particular, $a_1 b_1 = 0$. Then we have either $a_1 = 0$ or $b_1 = 0$. If $a_1 = 0$, then $sa = 0$. If $b_1 = 0$, then $sb = 0$. Thus, (0) is an S-primary ideal. Therefore, (0) is S-primary decomposable.*

Recall from [2], an ideal I of the ring R is called irreducible if from $I = J \cap K$ for some ideals J, K of R, it follows that either $I = J$ or $I = K$. It is well known that the classical proof of existence of primary decomposition in a Noetherian ring involves the concept of irreducible ideals. So we need an S-version of irreducible ideals to prove the existence of S-primary decomposition in S-Noetherian rings.

Definition 4.7. *An ideal Q (disjoint from S) of the ring R is called S-*irreducible *if from $s(I \cap J) \subseteq Q \subseteq I \cap J$ for some $s \in S$ and some ideals I, J of R, it follows that there exists $s' \in S$ such that either $ss'I \subseteq Q$ or $ss'J \subseteq Q$.*

It is clear from the definition that every irreducible ideal is an$S$-irreducible

ideal. However, the following example shows that an S-irreducible ideal need not be

irreducible.

Example 4.19. *Let $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus 3\mathbb{Z}$ and $I = 6\mathbb{Z}$. Since $I = 2\mathbb{Z} \cap 3\mathbb{Z}$, I is not an irreducible ideal of R. Now take $s = 2 \in S$. Then $2(3\mathbb{Z}) = 6\mathbb{Z} \subseteq I$. Thus, I is an S-irreducible ideal of R. Recall from [14], an ideal Q (disjoint from S) of a ring R is said to be S-prime if there exists $s \in S$ such that for $a, b \in R$ with $ab \in Q$, we have either $sa \in Q$ or $sb \in Q$. Clearly, every S-prime ideal is S-primary. The following example shows that the converse of this is not true in general.*

Example 4.20. *Consider $R = \mathbb{Z}$, $Q = 4\mathbb{Z}$, and $S = \mathbb{Z} \setminus 2\mathbb{Z}$. Notice that $4 \in Q$ but $2s \notin Q$ for all $s \in S$. This implies that Q is not an S-prime ideal of R. Obviously, Q is a primary ideal of R and hence S-primary.*

Recall from Proposition 2.5 of [23] that if Q is an S-primary ideal of a ring R, then $P = \mathrm{rad}(Q)$ is an S-prime ideal. In such a case, we say that Q is a P-S-primary ideal of R.

Proposition 4.11. *Let S be a multiplicative set of a ring R. Then the following statements hold:*

1. *Finite intersection of P-S-primary ideals is P-S-primary.*
2. *If Q is a P-primary ideal of R with $Q \cap S = \varnothing$, then for any ideal J of R with $J \cap S \neq \varnothing$, $Q \cap J$ is a $(P \cap \mathrm{rad}(J))$-S-primary ideal of R.*

Proof. (1) Let $Q_1, Q_2, \ldots, Q_n$ be P-S-primary ideals, then $S \cap Q_i = \emptyset$ for each $i = 1, 2, \ldots, n$, and so $S \cap (\bigcap_{i=1}^{n} Q_i) = \emptyset$. Suppose $Q = \bigcap_{i=1}^{n} Q_i$. Since each Q_i is P-S-primary, $\mathrm{rad}(Q) = \mathrm{rad}(\bigcap_{i=1}^{n} Q_i) = \bigcap_{i=1}^{n} \mathrm{rad}(Q_i) = P$. Now, let $xy \in Q$, where $x, y \in R$ and with $sy \notin Q$ for all $s \in S$. Consequently, for every $s \in S$, there exists k_s such that $xy \in Q_{k_s}$ and $sy \notin Q_{k_s}$. Let $s_i \in S$ be the element satisfying the S-primary property for Q_i. Since we have finitely many Q_i's, put $s = s_1 s_2 \cdots s_n \in S$. Now fix s and assume that $xy \in Q$ but $sy \notin Q$. Thus, there exists k such that $xy \in Q_k$ and $sy \notin Q_k$. Then for $s_k \in S$ we obtain $s_k x \in \mathrm{rad}(Q_k) = P$ or $s_k y \in Q_k$. The

latter case gives $sy \in Q_k$, a contradiction. Thus, $sx \in \text{rad}(Q) = P$, and therefore Q is P-S-primary.

(2) As $Q \cap S = \emptyset$, it follows that $(Q \cap J) \cap S = \emptyset$. By assumption, $J \cap S \neq \emptyset$. Let $s \in J \cap S$. Let $a, b \in R$ be such that $ab \in Q \cap J$. Either $a \in Q$ or $b \in \text{rad}(Q) = P$, since Q is a P-primary ideal of R. Hence, either $sa \in Q \cap J$ or $sb \in P \cap J \subseteq P \cap \text{rad}(J) = \text{rad}(Q) \cap \text{rad}(J) = \text{rad}(Q \cap J)$. This proves that $Q \cap J$ is a $(P \cap \text{rad}(J))$-S-primary ideal of R.

□

If an ideal I of a ring R admits an S-primary decomposition, then we prove in Remark 4.1 that it will admit a minimal S-primary decomposition.

Following [13], let E be a family of ideals of a ring R. An element $I \in E$ is said to be an S-maximal element of E if there exists $s \in S$ such that for each $J \in E$, if $I \subseteq J$, then $sJ \subseteq I$. Also an increasing sequence $(I_i)_{i \in \mathbb{N}}$ of ideals of R is called S-stationary if there exist a positive integer k and $s \in S$ such that $sI_n \subseteq I_k$ for all $n \geq k$.

The following theorem provides a connection between the concepts of S-irreducible ideals and S-primary ideals.

Theorem 4.8. *Let R be an S-Noetherian ring. Then every S-irreducible ideal of R is S-primary.*

Proof. Suppose Q is an S-irreducible ideal of R. Let $a, b \in R$ such that $ab \in Q$ and $sb \notin Q$ for all $s \in S$. Our aim is to show that there exists $t \in S$ such that $ta \in \text{rad}(Q)$.

Consider $A_n = \{\, x \in R : a^n x \in Q \,\}, \qquad n \in \mathbb{N}$. Clearly each A_n is an ideal of R and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an increasing chain of ideals of R. Since R is S-Noetherian, by Theorem 2.3 of [**?**], this chain is S-stationary; that is, there exist $k \in \mathbb{N}$ and $s \in S$ such that $sA_n \subseteq A_k \quad$ for all $n \geq k$.

Consider the two ideals $I = (a^k) + Q \quad$ and $\quad J = (b) + Q$ of R. Then $Q \subseteq I \cap J$.

For the reverse containment, let $y \in I \cap J$. Write $y = a^k z + q$ for some $z \in R$ and $q \in Q$. Since $ab \in Q$, we have $aJ \subseteq Q$, whence $ay \in Q$.

Now $a^{k+1}z = a(a^kz) = a(y-q) \in Q$. This implies that $z \in A_{k+1}$, and so $sz \in sA_{k+1} \subseteq A_k$. Consequently, $a^ksz \in Q$, which implies that $a^ksz + sq = sy \in Q$. Thus, we have $s(I \cap J) \subseteq Q \subseteq I \cap J$.

Since Q is S-irreducible, there exists $s' \in S$ such that either $ss'I \subseteq Q$ or $ss'J \subseteq Q$. If $ss'J \subseteq Q$, then $ss'b \in Q$, which is not possible. Therefore $ss'I \subseteq Q$, which implies that $ss'a^k \in Q$.

Put $t = ss' \in S$. Then $(ta)^k \in Q$, and hence $ta \in \mathrm{rad}(Q)$, as desired. □

Now we are in a position to prove the existence of S-primary decomposition in S-Noetherian rings as our main result.

Theorem 4.9. *(**Existence of S-primary decomposition**) Let R be an S-Noetherian ring. Then every proper ideal of R disjoint from S can be written as a finite interFor an ideal I of R, let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal S-primary decomposition, where Q_i is P_i-S-primary. Then the S-prime ideals P_i are said to belong to I.*

Also we say that P_i $(1 \leq i \leq n)$ is isolated S-prime *if $sP_j \nsubseteq P_i$ for all $s \in S$ and for all $j \neq i$. Other S-prime ideals are called* embedded S-prime ideals.*section of S-primary ideals.*

Proof. Let E be the collection of ideals of R which are disjoint from S and cannot be written as a finite intersection of S-primary ideals. We wish to show $E = \emptyset$.

On the contrary, suppose $E \neq \emptyset$. Since R is an S-Noetherian ring, by Theorem 2.3 of [6], there exists an S-maximal element in E, say I. Evidently, I is not an S-primary ideal, by Theorem 2.1, I is not an S-irreducible ideal, and so I is not an irreducible ideal. This implies that $I = J \cap K$ for some ideals J and K of R with $I \neq J$ and $I \neq K$. Since I is not S-irreducible, $sJ \nsubseteq I$ and $sK \nsubseteq I$ for all $s \in S$.

Now we claim that $J, K \notin E$. For this, if J (respectively, K) belongs to E, then since I is an S-maximal element of E and $I \subset J$ (respectively, $I \subset$

K), there exists s' (respectively, s'') from S such that $s'J \subseteq I$ (respectively, $s''K \subseteq I$). This is not possible as I is not S-irreducible. Therefore $J, K \notin E$.

This implies that J and K can be written as a finite intersection of S-primary ideals. Consequently, I can also be written as a finite intersection of S-primary ideals since $I = J \cap K$, a contradiction as $I \in E$. Thus, $E = \emptyset$, i.e., every proper ideal of R disjoint from S can be written as a finite intersection of S-primary ideals.

□

Recall that if I is any ideal of R, the radical of I is $\text{rad}(I) = \{x \in R : x^n \in I \text{ for some } n > 0\}$. If $I = \text{rad}(I)$, then I is called a radical ideal

Corollary 4.4. *Let R be an S-Noetherian ring. Then every radical ideal I (disjoint from S) of R is the intersection of finitely many S-prime ideals.*

Proof. By Theorem 4.9, there are finitely many S-primary ideals

$Q_1, Q_2, \ldots, Q_r$ such that $I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$. Also by Proposition 2.5 of [23], $\text{rad}(Q_i) = P_i$ is S-prime for $i = 1, 2, \ldots, r$. Consequently, $I = \text{rad}(I) = P_1 \cap P_2 \cap \cdots \cap P_r$. This completes the proof.

□

Remarks 4.1. *Let R be a ring, and let S be a multiplicative set of R. Let I be an ideal of R such that $I \cap S = \varnothing$. Suppose that I admits an S-primary decomposition. (In Theorem 4.9, we have shown that if R is S-Noetherian, then any ideal I of R disjoint from S admits an S-primary decomposition.) Let $I = \bigcap_{i=1}^{n} Q_i$ be an S-primary decomposition of I with Q_i being a P_i-S-primary ideal of R for each $i \in \{1, 2, \ldots, n\}$. Then, by Proposition 2.7 of [?], $S^{-1}Q_i$ is $S^{-1}P_i$-primary and $S(I) = \bigcap_{i=1}^{n} S(Q_i)$ is a primary decomposition with $S(Q_i)$ being an $S(P_i)$-primary ideal for each $i \in \{1, \ldots, n\}$.*

Let $1 \leq i \leq n$. Let $s_i \in S$ be such that it satisfies the S-primary property of Q_i. Notice that $(P_i : s_i)$ is a prime ideal of R and $(Q_i : s_i)$ is

a $(P_i : s_i)$*-primary ideal of* R*. Let* $s = \prod_{i=1}^{n} s_i$*. Then* $s \in S$*. Observe that* $(Q_i : s_i) = (Q_i : s)$, $(P_i : s_i) = (P_i : s)$, $S(Q_i) = (Q_i : s)$, $S(P_i) = (P_i : s)$.

From $I = \bigcap_{i=1}^{n} Q_i$*, it follows that* $(I : s) = \bigcap_{i=1}^{n}(Q_i : s) = \bigcap_{i=1}^{n} S(Q_i) = S(I)$*. Let* k *of the* $S(P_1), \ldots, S(P_n)$ *be distinct. After a suitable rearrangement of* $\{1, \ldots, n\}$*, we may assume without loss of generality that* $S(P_1), \ldots, S(P_k)$ *are distinct among* $S(P_1), \ldots, S(P_n)$*. Let*

$A_1 = \{\, j \in \{1, \ldots, n\} : S(Q_j)$ *is* $S(P_1)$*-primary* $\}$,..... $A_k = \{\, j \in \{1, \ldots, n\} : S(Q_j)$ *is* $S(P_k)$*-primary* $\}$.

From the above discussion, it is evident that $1 \in A_1, \ldots, k \in A_k$*, and* $\{1, \ldots, n\} = \bigcup_{t=1}^{k} A_t$*. Let* $1 \leq t \leq k$*. Notice that* $\bigcap_{j \in A_t} S(Q_j)$ *is* $S(P_t)$*-primary by Lemma 4.3 of [?]. It is convenient to denote* $\bigcap_{j \in A_t} Q_j$ *by* I'_t*. Thus,* $(I : s) = S(I) = \bigcap_{t=1}^{k} S(I'_t)$*, with* $S(I'_t)$ *being* $S(P_t)$*-primary for each* $t \in \{1, \ldots, k\}$ *and* $S(P_1), \ldots, S(P_k)$ *distinct. After omitting those* $S(I'_t)$ *such that* $S(I'_t) \supseteq \bigcap_{\substack{t \in \{1,\ldots,k\} \\ t \neq i}} S(I'_t)$*, we may assume without loss of generality that* $(I : s) = S(I) = \bigcap_{t=1}^{k} S(I'_t)$ *is a minimal primary decomposition of* $S(I)$*.*

Next, we claim that $I = (I : s) \cap (I + Rs)$*. It is clear that* $I \subseteq (I : s) \cap (I + Rs)$*. Let* $y \in (I : s) \cap (I + Rs)$*. Then* $ys \in I$ *and* $y = a + rs$ *for some* $r \in R$*. This implies that* $ys = as + rs^2$*, and so* $rs^2 \in I$*. Hence* $r \in S(I) = (I : s)$*. Therefore,* $y = a + rs \in I$*. Thus* $(I : s) \cap (I + Rs) \subseteq I$*, and hence* $I = (I : s) \cap (I + Rs) = \left(\bigcap_{t=1}^{k} S(I'_t)\right) \cap (I + Rs) = \bigcap_{t=1}^{k} \big(S(I'_t) \cap (I + Rs)\big)$*. For convenience, let us denote* $Q'_t = S(I'_t) \cap (I + Rs)$*. As* $S(I'_t)$ *is* $S(P_t)$*-primary with* $S(I'_t) \cap S = \varnothing$ *and* $(I + Rs) \cap S \neq \varnothing$*, it follows from Proposition 2.1 [2] that* Q'_t *is* $S(P_t) \cap \operatorname{rad}(I + Rs)$*-*$S$*-primary. Since* $S(S(P_t)) = S(P_t)$ *and* $S(\operatorname{rad}(I + Rs)) = R$*, we obtain* $S\big(S(P_t) \cap \operatorname{rad}(I + Rs)\big) = S(P_t)$*. Notice that* $S(Q'_t) = S(S(I'_t) \cap (I + Rs)) = S(I'_t)$*, as* $S(S(I'_t)) = S(I'_t)$ *and* $S(I + Rs) = R$*. Hence, for all distinct* $i, j \in \{1, \ldots, k\}$*,* $S(S(P_i) \cap \operatorname{rad}(I + Rs)) \neq S(S(P_j) \cap \operatorname{rad}(I + Rs))$*, and for each* i*,* $S(Q'_i) \nsubseteq \bigcap_{\substack{t \in \{1,\ldots,k\} \\ t \neq i}} S(Q'_t)$*. Therefore,* $I = \bigcap_{t=1}^{k} Q'_t$ *is a minimal* S*-primary decomposition.*

Recall from [?], let R *be a ring,* $S \subseteq R$ *a multiplicative set and* I *an*

ideal of R disjoint from S. Let P be an S-prime ideal of R such that $I \subseteq P$. Then P is said to be a minimal S-prime ideal over I *if P is minimal in the set of the S-prime ideals containing I.*

Also in Remark 2 of [?] proves that in S-Noetherian rings, the set of minimal S-prime ideals is finite if S is a finite multiplicative set.

A prime ideal is said to be a minimal prime ideal if it is a minimal prime ideal over the zero ideal. Noether showed that in a Noetherian ring, there are only finitely many minimal prime ideals over any given ideal, see [17], Theorem 88.

Example 4.21. *Consider the ring $R = F[x_1, x_2, \ldots, x_n, \ldots]/(x_i x_j : i \neq j, i, j \in \mathbb{N})$, where F is a field. Let $y_i = \overline{x_i}$ be the image of x_i under the canonical map. Consider the multiplicative set $S = \{y_1^n : n \in \mathbb{N} \cup \{0\}\}$. Then $(y_1) \subseteq (y_1, y_2) \subseteq \cdots \subseteq (y_1, y_2, \ldots, y_n) \subseteq \cdots$ is an ascending chain of ideals of R which is not stationary. Consequently, R is not a Noetherian ring. Evidently, R is an S-Noetherian ring, see [26], Example 4. Note that if $P \subset R$ is a prime ideal, then there can be at most one i such that $y_i \notin P$. It follows that if P_i is the ideal generated by all the y_j for $j \neq i$, each P_i is prime since $R/P_i \cong F[x_i]$. Clearly, every P_i is a minimal prime ideal of R. Thus, R has infinitely many minimal prime ideals.*

Remarks 4.2. *Let R be a ring, $S \subseteq R$ be a multiplicative set, and $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ be a minimal S-primary decomposition, where Q_i is P_i-S-primary ideal of R. If $Z(R/P_i) \cap \overline{S_i} = \emptyset$, where $\overline{S_i} = \{s + P_i : s \in S\}$ and $Z(R/P_i)$ denotes the set of zero divisors of R/P_i for all $i = 1, 2, \ldots, r$, then each P_i is a prime ideal of R. For this we will show that $(P_i : s) = P_i$ for all $s \in S$ and $i = 1, 2, \ldots, r$. It is clear that $P_i \subseteq (P_i : s)$. Conversely, let $s \in S$ and $x \in (P_i : s)$, then $sx \in P_i$; so $(s + P_i)(x + P_i) = P_i$. Thus, $x \in P_i$ since $Z(R/P_i) \cap \overline{S_i} = \emptyset$. Now, if P_i is an S-prime ideal of R, then by Proposition 1 of [14], $(P_i : s_i)$ is prime ideal of R for some $s_i \in S$. Since $P_i = (P_i : s_i)$, P_i is a prime ideal of R.*

Theorem 4.10. *Let R be an S-Noetherian ring, $S \subseteq R$ a multiplicative set, and $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ be a minimal S-primary decomposition, where Q_i is P_i-S-primary ideal of R. If $Z(R/P_i) \cap \overline{S_i} = \emptyset$, then minimal prime ideals of R are in the set $\{P_i : i = 1, 2, \ldots, r\}$.*

Proof. Since $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_r$, $\text{rad}(0) = P_1 \cap P_2 \cap \cdots \cap P_r$, where $P_i = \text{rad}(Q_i)$ $(1 \leq i \leq r)$ is an S-prime ideal of R. By Remark 4.2, each P_i $(1 \leq i \leq r)$ is a prime ideal of R. Let P' be any minimal prime ideal of R. Then $(0) \subseteq P'$, and so $\text{rad}(0) \subseteq \text{rad}(P') = P'$. This implies $P_1 \cap P_2 \cap \cdots \cap P_r \subseteq P'$. By Proposition 1.11(ii) of [2], there exists $j \in \{1, 2, \ldots, r\}$ such that $P_j \subseteq P'$. Since P' is a minimal prime ideal, $P' = P_j$, as desired.

□

Proposition 4.12. *Let $Q_1, Q_2, \ldots, Q_n$ be ideals of R, and P be an S-prime ideal containing $\bigcap_{k=1}^{n} Q_k$. Then there exists $s \in S$ such that $sQ_k \subseteq P$ for some k. In particular, if $P = \bigcap_{k=1}^{n} Q_k$, then $sQ_k \subseteq P \subseteq Q_k$ for some k.*

Proof. On the contrary, suppose $sQ_k \nsubseteq P$ for all $s \in S$ and $k = 1, 2, \ldots, n$. Then for each k there exists $x_k \in Q_k$ such that $sx_k \notin P$ for all $s \in S$. Evidently, $x_1 x_2 \cdots x_n \in Q_1 Q_2 \cdots Q_n \subseteq \bigcap_{k=1}^{n} Q_k \subseteq P$. Since P is an S-prime ideal, by Proposition 4 of [14], there exists $s' \in S$ and $j \in \{1, 2, \ldots, n\}$ such that $s'x_j \in P$, a contradiction. Hence, there exist $s \in S$ and $k \in \{1, 2, \ldots, n\}$ such that $sQ_k \subseteq P$. Finally, if $P = \bigcap_{k=1}^{n} Q_k$, then by the above argument there exist $s \in S$ and $k \in \{1, 2, \ldots, n\}$ such that $sQ_k \subseteq P \subseteq Q_k$.

□

Now we prove the S-version of the 1st uniqueness theorem:

Theorem 4.11. *(**S-version of 1st S-uniqueness theorem**). Let R be a ring and let S be a multiplicative set of R. Let I be an ideal of R which*

admits S-primary decomposition. Let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal S-primary decomposition, where Q_i is P_i-S-primary for each $i \in \{1, 2, \ldots, n\}$. Then the $S(P_i)$ are precisely the prime ideals which occur in the set of ideals $S(\mathrm{rad}(I : x))$ $(x \in R)$, and hence are independent of the particular S-decomposition of I

Proof. For any $x \in R$ we have $S^{-1}(I : x) = \left(\bigcap_{i=1}^{n} S^{-1}Q_i :_{S^{-1}R} \frac{x}{1}\right) = \bigcap_{i=1}^{n} \left(S^{-1}Q_i :_{S^{-1}R} \frac{x}{1}\right)$, and so $S(I : x) = \bigcap_{i=1}^{n}(S(Q_i) : x)$. Evidently, $S(\mathrm{rad}(I : x)) = \bigcap_{i=1}^{n} \mathrm{rad}(S(Q_i) : x)$.

Notice that each $S(P_i)$ is a prime ideal since $S^{-1}P_i$ is a prime ideal and contraction of a prime ideal is a prime ideal. Now we show that each $S(Q_i)$ is $S(P_i)$-primary. For this, let $x, y \in R$ such that $xy \in S(Q_i)$. Then $xy/1 \in S^{-1}Q_i$, $xy/1 = a/s$ for some $a \in Q_i$ and $s \in S$. Consequently, there exists $u \in S$ such that $usxy \in Q_i$. Since Q_i is P_i-S-primary, there exists $t \in S$ such that either $tusx \in Q_i$ or $ty \in P_i$. It follows that either $x/1 = tusx/(tus) \in S^{-1}Q_i$ or $y/1 = ty/t \in S^{-1}P_i$. This implies that either $x \in S(Q_i)$ or $y \in S(P_i)$, and $\mathrm{rad}\, S(Q_i) = S(\mathrm{rad}(Q_i)) = S(P_i)$. Therefore, $S(Q_i)$ is $S(P_i)$-primary.

Then $S(\mathrm{rad}(I : x)) = \bigcap_{x \notin S(Q_i)} S(P_i)$, by Lemma 4.4 of [2]. Suppose $S(\mathrm{rad}(I : x))$ is prime; then we have $S(\mathrm{rad}(I : x)) = S(P_j)$ for some j, by Proposition 1.11 of [2]. Hence, every prime ideal of the form $S(\mathrm{rad}(I : x))$ is one of the $S(P_i)$. Conversely, for each i there exists

$$x_i \in \left(\bigcap_{j \in \{1,2,\ldots,n\} \setminus \{i\}} S(Q_j) \right) \setminus S(Q_i)$$

since the S-decomposition is minimal. Consequently, for each i, we have $S(I : x_i) = \bigcap_{j=1}^{n}(S(Q_j) : x_i) = (S(Q_i) : x_i)$ since $(S(Q_j) : x_i) = R$ for all $j \neq i$, by Lemma 4.4 of [2]. Then $S(\mathrm{rad}(I : x_i)) = S(P_i)$, by Lemma 4.4 of [2]. This completes the proof. □

Lemma 4.5. *Let R be a ring, $S \subseteq R$ be a multiplicative set, and Q be an P-S-primary ideal of R. Then $S^{-1}Q$ is $S^{-1}P$-primary and its contraction in R satisfies $tS(Q) \subseteq Q \subseteq S(Q)$ for some $t \in S$.*

Proof. Clearly, $S^{-1}Q \neq S^{-1}R$ since $Q \cap S = \varnothing$ and $P = \text{rad}(Q)$. Let $x \in S(Q)$. Then $x/1 = a/s$ for some $a \in Q$ and $s \in S$. Consequently, there exists $s' \in S$ such that $(xs-a)s' = 0$; whence $ss'x = s'a \in Q$. As Q is S-primary, there exists $t \in S$ such that either $ss't \in \text{rad}(Q) = P$ or $ta \in Q$. This implies that $tx \in Q$ since $P \cap S = \varnothing$, and so $tS(Q) \subseteq Q \subseteq S(Q)$. Now since Q is S-primary, by Proposition 2.7 of [23], $S^{-1}Q$ is a primary ideal of $S^{-1}R$ and $\text{rad}(S^{-1}Q) = S^{-1}(\text{rad}(Q)) = S^{-1}P$.

Also since P is an S-prime ideal of R, by Remark 1 of [14], $S^{-1}P$ is a prime ideal of $S^{-1}R$. Thus, $S^{-1}Q$ is an $S^{-1}P$-primary ideal of $S^{-1}R$.

□

Lemma 4.6. *Let R be a ring and let S be a multiplicative set of R. Let I be an ideal of R such that I admits an S-primary decomposition. Let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal S-primary decomposition, where Q_i is P_i-S-primary for each $i \in \{1, 2, \ldots, n\}$. Let S' be a multiplicative set of R such that $S \subseteq S'$. If $Q_1, Q_2, \ldots, Q_m$ are such that $Q_i \cap S' = \varnothing$ for each $i \in \{1, \ldots, m\}$ and $Q_i \cap S' \neq \varnothing$ for each $i \in \{m+1, \ldots, n\}$, then*

$$S'^{-1}I = \bigcap_{i=1}^{m} S'^{-1}Q_i$$

is a primary decomposition and there exists $t \in S$ such that $tS'(I) \subseteq \bigcap_{i=1}^{m} Q_i \subseteq S'(I)$.

Proof. By Proposition 3.11 (v) of [2], $S'^{-1}I = \bigcap_{i=1}^{n} S'^{-1}Q_i$. Since $S' \cap Q_i \neq \varnothing$ for $i = m+1, \ldots, n$, by Proposition 4.8 (i) of [2], $S'^{-1}Q_i = S'^{-1}R$ for $i = m+1, \ldots, n$. Consequently, $S'^{-1}I = \bigcap_{i=1}^{m} S'^{-1}Q_i$. As Q_i is P_i-S-primary and $S \subseteq S'$, it follows that Q_i is P_i-S'-primary for each $i = 1, \ldots, m$. Also by Lemma 4.5, $S'^{-1}Q_i$ is $S'^{-1}P_i$-primary for $i = 1, \ldots, m$. Thus $S'^{-1}I = \bigcap_{i=1}^{m} S'^{-1}Q_i$ is a primary decomposition.

Next, let $x \in S'(I)$. Then $x/1 \in S'^{-1}I = \bigcap_{i=1}^{m} S'^{-1}Q_i$. Write $x/1 = a_i/s_i$ for some $a_i \in Q_i$ and $s_i \in S'$. This implies that $(xs_i - a_i)s = 0$ for some $s \in S'$; whence $s_i sx = sa_i \in Q_i$ for $i = 1, \ldots, m$. Then there exists $t_i \in S$ such that $t_i x \in Q_i$ since Q_i is P_i-S-primary and $P_i \cap S' = \varnothing$

for all $i = 1, \ldots, m$. Put $t = t_1 t_2 \cdots t_m$. Then $tx \in \bigcap_{i=1}^{m} Q_i$, and hence $tS'(I) \subseteq \bigcap_{i=1}^{m} Q_i \subseteq S'\Big(\bigcap_{i=1}^{n} Q_i\Big) = \bigcap_{i=1}^{n} S'(Q_i) = S'(I)$. Hence, $tS'(I) \subseteq \bigcap_{i=1}^{m} Q_i \subseteq S'(I)$.

□

For an ideal I of R, let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal S-primary decomposition, where Q_i is P_i-S-primary. Then the S-prime ideals P_i are said to belong to I.

Also we say that P_i $(1 \leq i \leq n)$ is *isolated S-prime* if $sP_j \nsubseteq P_i$ for all $s \in S$ and for all $j \neq i$. Other S-prime ideals are called *embedded S-prime ideals*.

Theorem 4.12. ***(S-version of 2nd uniqueness theorem)***. *Let I be an ideal of R which admits S-primary decomposition, that is, $I = \bigcap_{i=1}^{n} Q_i$ be a minimal S-primary decomposition of I, where Q_i is P_i-S-primary. If $\{P_1, \ldots, P_m\}$ is a set of isolated S-prime ideals of I for some m $(1 \leq m \leq n)$, then $Q_1 \cap \cdots \cap Q_m$ is independent of S-primary decomposition.*

Proof. Since each P_i is S-prime, by Proposition 1 of [14], there exists $s_i \in S$ such that $(P_i : s_i)$ is a prime ideal for $i = 1, \ldots, m$. Consider the multiplicative set $S' = R - \bigcup_{i=1}^{m}(P_i : s_i)$. Evidently, $S' \cap P_i = \varnothing$ for $i = 1, \ldots, m$ since $P_i \subseteq (P_i : s_i)$. We claim that $S' \cap P_k \neq \varnothing$ for $k = m+1, \ldots, n$. On the contrary, $S' \cap P_k = \varnothing$ for some k. This implies that $P_k \subseteq \bigcup_{i=1}^{m}(P_i : s_i)$, by Proposition 1.11 (i) of [2], $P_k \subseteq (P_j : s_j)$ for some j $(1 \leq j \leq m)$. Consequently, $s_j P_k \subseteq P_j$, a contradiction since P_j is an isolated S-prime. This concludes that $S' \cap \big(\bigcup_{i=1}^{m} Q_i\big) = \varnothing$ and $S' \cap \big(\bigcap_{i=m+1}^{n} Q_i\big) \neq \varnothing$ since $P_i = \mathrm{rad}(Q_i)$ for all $i = 1, \ldots, n$.

Now since each P_i is S-prime, $S \cap P_i = \varnothing$ for $i = 1, \ldots, m$. This implies that $S \cap (P_i : s_i) = \varnothing$ for all $i = 1, \ldots, m$. For, if $S \cap (P_i : s_i) \neq \varnothing$ for some i, then there exists $s' \in S$ such that $s's_i \in P_i$, a contradiction. Consequently, $S \subseteq R - \bigcup_{i=1}^{m}(P_i : s_i) = S'$.

Also since each Q_i is S-primary and $S' \cap Q_i = \varnothing$ for all $i = 1, \ldots, m$, Q_i is S'-primary for all $i = 1, \ldots, m$. Thus, by Lemma 4.6, $tS'(I) \subseteq \bigcap_{i=1}^{m} Q_i \subseteq S'(I)$ for some $t \in S$. From the above it is clear that $Q_1 \cap \cdots \cap Q_m$

depends only on I, and hence $Q_1 \cap \cdots \cap Q_m$ is independent of S-primary decomposition.

□

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